(1) \(f\) has an interior discontinuity at \(x = -\frac{1}{2}\).
\(f\) is piecewise smooth.

(b) Because of the discontinuities, the convergence will be like \(1/n\).

(c) The sine series represents the periodic extension of the odd extension of \(f\).

Again discontinuities and \(1/n\) convergence.

(d) Here we have the periodic extension of the even extension. We see that the extended function is continuous but the extended derivative is not, so the convergence is like \(1/n^2\).

(2) (a) The weight function is \(\delta(x)\), so \(\int_{-\infty}^{\infty} \delta(x) \, dx = 0\).

(b) We construct the Rayleigh quotient.

\[ \psi(k \psi')' + \lambda \psi^2 = 0 \]
\[ (\psi k \psi')' - k \psi^2 + \lambda \psi^2 = 0. \]
(2) (b) continued. We integrate over \([0,1]\) and use the boundary conditions to get
\[
\lambda = \frac{\int_0^1 kw'^2 \, dx}{\int_0^1 kw^2 \, dx} > 0.
\]
This shows that \(\lambda\) is non-negative. For \(\lambda\) to be zero, we would have to have \(w' = 0\), hence \(w = \text{constant}\), but if \(w(1) = 0\) \(\Rightarrow \text{constant} = 0\) \(\Rightarrow \lambda > 0\).

(c) The \(\psi_n\)'s are appropriate eigenfunctions for this problem, so we try
\[
T(x,t) = \sum_{n=1}^{\infty} c_n(t) \psi_n(x).
\]
We substitute into the equation to get
\[
\sigma(x) \sum_{n=1}^{\infty} \frac{d}{dt} c_n(t) \psi_n(x) = \sum_{n=1}^{\infty} c_n(t) \frac{d}{dx}(x \frac{d\psi_n}{dx})
\]
so
\[
\sum_{n=1}^{\infty} \frac{d}{dt} c_n(t) \psi_n(x) = -\sigma(x) \sum_{n=1}^{\infty} \lambda_n c_n(t) \psi_n(x).
\]
We choose coefficients to get
\[
\frac{d c_n}{dt} = -\lambda_n c_n(t)
\]
and
\[
c_n(t) = c_n(0) e^{-\lambda_n t}.
\]
But
\[
T(x,0) = f(x) = \sum_{n=1}^{\infty} c_n(0) \psi_n(x),
\]
so
\[
c_n(0) = \frac{\int_0^1 \sigma(x) \psi_n(x) f(x) \, dx}{\int_0^1 \sigma(x)[\psi_n(x)]^2 \, dx}.
\]
If you described this process carefully rather than carrying it out, you will get full credit.
(3) The solutions will be oscillatory in $y$, and we can see from the boundary conditions that the appropriate $y$-functions are $\sin \left( \frac{\pi y}{L} \right)$. We try

$$\Phi(x, y) = \sum_{n=1}^{\infty} a_n(x) \sin \left( \frac{n\pi y}{L} \right).$$

We substitute this in the equation to get

$$\sum_{n=1}^{\infty} \frac{d^2}{dx^2} a_n(x) \sin \left( \frac{n\pi y}{L} \right) + \sum_{n=1}^{\infty} \left( -\frac{n^2 \pi^2}{L^2} \right) a_n \sin \left( \frac{n\pi y}{L} \right) = 0.$$

We balance coefficients: $\frac{d^2}{dx^2} a_n(x) - \frac{n^2 \pi^2}{L^2} a_n = 0$.

The solution which vanishes at $x = 0$ is $a_n \sin \left( \frac{n\pi x}{L} \right)$. Then

$$\Phi(x, y) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi y}{L} \right).$$

We impose the inhomogeneous BC at $x = a$:

$$\frac{d\Phi}{dx}(a, y) = \sum_{n=1}^{\infty} \frac{\pi}{L} a_n \cos \left( \frac{n\pi a}{L} \right) \sin \left( \frac{n\pi y}{L} \right) = \Phi_0 \sin \left( \frac{\pi y}{L} \right).$$

We balance coefficients to get $a_n = 0$, $n \neq 3$ and

$$\frac{3\pi}{L} a_3 \cos \left( \frac{3\pi a}{L} \right) = \Phi_0.$$

So

$$a_3 = \frac{\Phi_0}{(3\pi a) \cos \left( \frac{3\pi a}{L} \right)}.$$

And

$$\Phi(x, y) = \Phi_0 \frac{\sin \left( \frac{3\pi x}{L} \right)}{(3\pi a \cos \left( \frac{3\pi a}{L} \right)) \sin \left( \frac{3\pi y}{L} \right)}.$$
We transform the equation using the first hint.
\[ \frac{d^2 \tilde{\phi}}{dy^2} - k^2 \tilde{\phi} = \pi e^{-\frac{1}{4k^2}} e^{-y}. \]

We look for a particular solution \( \tilde{\phi}_p = Ce^{-y} \):
\[ Ce^{-y} - k^2 e^{-y} = \pi e^{-\frac{1}{4k^2}} e^{-y} \]
\[ C = \frac{\pi e^{-\frac{1}{4k^2}}}{1 - k^2}. \]

The solution of the homogeneous equation is \( A e^{\frac{1}{2}ky} + Be^{-\frac{1}{2}ky} \). The solution is then
\[ \tilde{\phi} = A e^{\frac{1}{2}ky} + B e^{-\frac{1}{2}ky} + \frac{\pi e^{-\frac{1}{4k^2}}}{1 - k^2} e^{-y}. \]

To satisfy the condition \( \phi(x_0) = 0 \), \( A = 0 \). The other condition is \( \frac{\partial \phi}{\partial y} \bigg|_{y=0} = 0 \), so \( B = \frac{\pi e^{-\frac{1}{4k^2}}}{k^2} \).

Then \( \tilde{\phi}(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2k^2}y} e^{ix} \left(e^{\frac{1}{2}ky} - e^{-\frac{1}{2}ky}\right)}{k^2 - 1} dk \).

Then \( \phi(r,\phi) = \sum_{n=0}^{\infty} C_n \frac{P_n(\cos \phi)}{r^{n+1}} \).

The boundary condition is \( \phi_0 \bigg|_{\theta} = \theta \big( \cos^2 \phi \big) \).

We have \( \lambda = P_0(\cos \phi) \) and \( b(\cos \phi) = \frac{b}{3} \left[ P_0 + 2P_2 \right] \)

Then \( \frac{\partial \phi}{\partial r}(r,\phi) = -\sum_{n=0}^{\infty} \int_{0}^{\pi} (n+1) C_n \frac{P_n'(\cos \theta)}{r^{n+2}} \frac{\sin \theta}{\sin \phi} = \frac{b}{6} \frac{1}{6} P_0 \frac{a}{a + \frac{2b}{3} P_2} \).

We believe coefficients: \( C_n = 0 \), for \( n \neq 0,2 \)
\[ -\frac{\cos}{a^2} - \frac{\cos}{a^2} \left( 1 + \frac{b}{3} \right) - \frac{3(1 + \frac{b}{3})}{a^2} = \frac{4b}{a} \quad \text{for } a = \frac{2b}{3} \]
\[ b = \frac{4b}{a} \frac{P_0(\cos \phi)}{1 + \frac{b}{3}} \frac{1}{r^2} \]
\[ \text{max rate of dropoff is for } b = -3. \]
We substitute the expansion into the equation:

\[ \sum_{n=1}^{\infty} \frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} J_0 (\alpha_n \frac{r}{a}) \right] \alpha_n (z) + \sum_{n=1}^{\infty} \frac{d^2 \alpha_n}{dz^2} J_0 (\alpha_n \frac{r}{a}) = 0. \]

From the hint we have

\[ \frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} J_0 (\alpha_n \frac{r}{a}) \right] = -\frac{\alpha_n^2}{a^2} J_0 (\alpha_n \frac{r}{a}). \]

Now we can balance coefficients to get

\[ \frac{d^2 \alpha_n}{dz^2} - \frac{\alpha_n^2}{a^2} \alpha_n = 0. \]

The solution is

\[ \alpha_n (z) = A_n e^{-\frac{\alpha_n z}{a}} + B_n e^{\frac{\alpha_n z}{a}}. \]

The second term blows up as \( z \to \infty \), so we require \( B_n = 0 \). Then

\[ \Phi (r, z) = \sum_{n=1}^{\infty} A_n e^{-\frac{\alpha_n z}{a}} J_0 (\alpha_n \frac{r}{a}). \]

Finally we impose the BC at \( z = 0 \):

\[ \Phi (r, 0) = \sum_{n=1}^{\infty} A_n J_0 (\alpha_n \frac{r}{a}) = f(0) = \sum_{n=1}^{\infty} f_n J_0 (\alpha_n \frac{r}{a}). \]

We balance coefficients to get \( A_n = f_n \).

Then

\[ \Phi (r, z) = \sum_{n=1}^{\infty} f_n e^{-\frac{\alpha_n z}{a}} J_0 (\alpha_n \frac{r}{a}). \]