ME201/MTH281/ME400/CHE400 FALL 2011
FINAL EXAM REVIEW

TIME AND PLACE OF EXAM

The exam will be on Wednesday Dec. 21, 7:15 – 10:15 PM, in Hoyt.

MATERIAL COVERED BY EXAM

The exam will cover all topics in the course, including homework assignments #1 - 11. The exam will be open book and open notes -- you may use any reference material you bring. Calculators are allowed during the exam but laptops are not.

OFFICE HOURS

There will be no TA office hours during the reading and exam periods. My office hours from Monday Dec. 12 until the exam are given below. Any changes to these will be posted on the web. I will be available by email (clark@me.rochester.edu) throughout this period, including evenings and weekends. On Monday through Friday, I will generally be in my office most of every week day. You are welcome to come in for questions at times other than the office hours, but you should email or call (275-4078) first to make sure that I will be there.

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REVIEW HELP

TUE DEC 20 REVIEW SESSION  CLARK BOWMAN  MOREY 321  12:30 – 2:30 PM

TUE DEC 20  Q&A SESSION  STEPHANIE C AI  CARLSON 2ND FLOOR  7:30 – 9:30 PM

PRACTICE FINAL EXAM

The final exam from 2010, with solutions, is available on the web. You should try working that exam in 3 hours after you have completed your review of the course material. The exam for 2011 will have 6 problems, and the scope will be similar to that of the 2010 exam.
SUGGESTED REVIEW PROBLEMS

You should work all of the review problems given here. Some of these problems are straightforward, and some are quite difficult. The most difficult ones are more difficult than the kinds of problems that will be on the final. However, taken together, these problems do cover the essential ideas that will be on the final. Solutions will not be handed out for these problems, but the answers for most problems are given. Any hints, corrections, or additions to the answers will be posted on the course web site.

PROBLEMS

(1) A vibrating string of length \( L \) is fixed at the ends \( x = 0 \) and \( x = L \). The string has a constant tension \( T \) and a spatially variable density \( \sigma(x) \). The vertical displacement of the string, \( y(x,t) \), is a solution of the problem given below.

\[
\sigma(x) \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2},
\]

\( y(0,t) = 0, \ y(L,t) = 0, \) and \( y(x,0) = f(x), \ \frac{\partial y(x,0)}{\partial t} = 0. \)

Express the solution \( y(x,t) \) in terms of the eigenfunctions \( \psi_n(x) \) of the Sturm-Liouville problem given below.

\[
\frac{d^2 \psi_n}{dx^2} + \lambda_n \sigma(x) \psi_n = 0, \ 0 < x < L, \text{ with } \psi_n(0) = 0 \text{ and } \psi_n(L) = 0.
\]

(2) Find the full Fourier series for the function \( f(x) = e^{-|x|} \) on the interval \(-1 \leq x \leq 1\). Does this series converge to \( f(x) \) everywhere on the interval? Relate the rate of convergence of the series to the smoothness of the periodically extended \( f(x) \).

(3) Find the Fourier sine series for \( f(x) = 1 - x \) on the interval \( 0 \leq x \leq 1 \). Sketch the periodically extended function represented by this series. Tell where the series converges to \( f(x) \) in the interval \([0,1]\), and discuss the rate of convergence of the series in relation to the smoothness of the periodically extended function.

(4) Solve the boundary value problem given below for the heat equation in a slab.

\[
\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} + \Gamma e^{-\alpha x}, \quad 0 < x < L \text{ and } t > 0, \text{ with } T(x,0) = 0, \ T(0,t) = 0, \text{ and } T(L,t) = T_1.
\]

Here \( D, \Gamma, \alpha, T_1 \) are all constant. Find a simple approximate solution, valid for large times, which shows the approach to steady state. (Don't spend a lot of time evaluating integrals, but make sure you understand the structure of the problem.)
(5) Solve the boundary value problem given below for the Laplace equation in a rectangle.

\[ \nabla^2 T = 0 \quad , \quad 0 < x < a , \quad 0 < y < b , \]

with \( \frac{\partial T}{\partial x}(0,y) = \frac{\partial T}{\partial x}(a,y) = 0, T(x,0) = 0, \) and \( T(x,b) = \gamma x, \) where \( \gamma \) is constant.

(6) Consider the regular Sturm-Liouville system given below for the interval \( 0 \leq x \leq 1. \) Find explicitly the eigenfunctions and eigenvalues. Expand the function \( f(x) = e^{-x} \) in a series of these functions.

\[
\frac{d}{dx} \left( e^{2x} \frac{dy}{dx} \right) + e^{2x} y + e^{2x} \lambda = 0 \quad , \quad 0 \leq x \leq 1 ,
\]

with \( y(0) = 0 \quad , \quad y(1) = 0 . \)

(7) Solve the boundary value problem given below for the Laplace equation in the semi-infinite region \( 0 \leq x \leq 2 , \ y \geq 0. \) Find a simple approximation to the solution valid for large \( y. \) (It is sufficient to write the coefficients as ratios of integrals. You do not have to evaluate the integrals.)

\[ \nabla^2 \Phi = 0 \ , \ 0 < x < 2 \ , \ y > 0 \quad , \quad \text{with } \Phi \to 0 \text{ as } y \to \infty , \]

\[ \Phi(0,y) = 0, \ \frac{\partial \Phi}{\partial x}(2,y) + 3\Phi(2,y) = 0, \ \text{and } \Phi(x,0) = x. \]

(8) Consider the boundary value problem given below for the two-dimensional Laplace equation in an infinite strip.

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad , \quad -\infty < x < \infty , \quad 0 < y < b \quad ,
\]

with \( \Phi(x,0) = \Phi(x,b) = \Phi_0 e^{-(x/a)^2} , \)

where \( \Phi_0 \) and \( a \) are constants. Solve this problem with the Fourier transform, and show that

\[ \Phi(x,y) = \frac{\Phi_0 a}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a^2 k^2} \left( \frac{\sinh(ky)}{\sinh(kb)} + \frac{\sinh(k[b-y])}{\sinh(kb)} \right) dk . \]

You do not need to evaluate this integral.

(9) Solve the boundary value problem below for the steady state temperature in a sphere of radius \( a. \) The source term \( \gamma \) and the parameter \( T_0 \) appearing in the boundary condition are both positive constants.

\[ \nabla^2 T = \gamma \ r < a \ , \text{with } T(a,\phi) = T_0 \cos(2\phi) . \]

(10) Solve the boundary value problem given below for radially symmetric transient heat conduction in a sphere.

\[ T = T(r,t), \ \text{with } \frac{\partial T}{\partial t} = D \nabla^2 T \ , \ r < a \ , \ T(a,t) = 0 \ , \text{and } T(r,0) = T_0 \ , \text{with } T_0 \text{ constant.} \]
(11) Consider axisymmetric waves in a circular cylinder of radius \( a \) and height \( h \). These waves are solutions of the wave equation

\[
\frac{\partial^2 \Psi}{\partial t^2} = c^2 \nabla^2 \Psi.
\]

For such waves \( \Psi = \Psi(r,z,t) \), with no dependence on the cylindrical angle \( \theta \). The boundary condition is that \( \Psi \) must vanish on the surface of the cylinder. Standing waves are normal modes of oscillation of the form \( \Psi(r,z,t) = \cos(\omega t) \Phi(r,z) \). Show that the spatial amplitude of such modes satisfies the Helmholtz equation \( \nabla^2 \Phi = -k^2 \Phi \), where \( k = \omega / c \). Use separation of variables to find all of the modes for the cylinder. Find the lowest frequency.

(12) Solve the boundary value problem given below for the Laplace equation in a semi-infinite cylinder.

\[
\nabla^2 \Phi(r,z) = 0, \quad 0 < r < a, \quad 0 < z < \infty,
\]

with \( \Phi(r,0) = \Phi_0 \left( 1 - \frac{r^2}{a^2} \right), \Phi(a,z) = 0, \) and \( \Phi \to 0 \) as \( z \to \infty \).

**ANSWERS AND HINTS FOR REVIEW PROBLEMS**

(1) Answer: \( y(x,t) = \sum_{n=1}^{\infty} A_n \cos(\omega_n t) \psi_n(x) \), where \( \omega_n = \sqrt{\lambda_n T} \).

The coefficients \( A_n \) are determined as usual by orthogonality and the initial values \( y(x,0) = f(x) \) and \( \partial y(x,0) / \partial t = 0 \).

\[
A_n = \frac{1}{N_n} \int_0^L f(x) \sigma(x) \psi_n(x) dx, \quad \text{where } N_n = \int_0^L \sigma(x) \left[ \psi_n(x) \right]^2 dx.
\]

(2) \( a_0 = 1 - e^{-1}, \quad a_n = \frac{2[1 - (-1)^n e^{-1}]}{1 + (n\pi)^2}, \quad b_n = 0 \).

(3) \( 1 - x = \sum_{n=1}^{\infty} \frac{2 \sin(n\pi x)}{n\pi}, \) for \( 0 < x \leq 1 \).

(4) The steady-state solution is \( T_s(x) = T_1 \frac{x}{L} + \frac{\Gamma_1}{Dx} \left[ 1 - e^{-ax} - \frac{x}{L} (1 - e^{-al}) \right] \).

An approximation valid for large times is \( T(x,t) \approx T_s(x) + C_1 \exp \left( -\frac{\pi^2 D t}{L^2} \right) \sin \left( \frac{\pi x}{L} \right) \),

where \( C_1 = -\frac{2}{L} \int_0^L T_s(x) \sin(\pi x / L) dx = -\frac{2T_1}{\pi} - \frac{2L^2 \Gamma(1 + e^{-al})}{\pi D(\pi^2 + L^2 a^2)} \).

(5) \( T(x,y) = \frac{\gamma a}{2b} y - \sum_{n=1}^{\infty} 4 \gamma a \frac{\sinh(n\pi y / a)}{\sinh(n\pi b / a)} \cos(n\pi x / a) \).

(6) The (non-normalized) eigenfunctions are \( y_n(x) = e^{-x} \sin(n\pi x) \). These functions are orthogonal with respect to the weight function \( e^{-x} \). The desired expansion is

\[
e^{-x} = \sum_{n=1}^{\infty} \frac{4}{n\pi} y_n(x).
\]

You should recognize this entire problem as a minor variation of the Fourier sine series.
(7) Solution by separation of variables leads to the following \( x \)-equation:

\[ F''(x) + \lambda F(x) = 0, \quad F(0) = 0, \quad F'(2) + 3F(2) = 0. \]

The resulting eigenvalue equation is \( \tan(z) = -z/6 \), where \( z = 2\sqrt{\lambda} \). Numerical analysis shows that the smallest eigenvalue is \( z_1 = 2.71646 \). By superposition we find the solution

\[
\Phi(x,y) = \sum_{i=1}^{\infty} C_i \exp\left(-\frac{z_i y}{2}\right) \sin\left(\frac{z_i x}{2}\right), \quad \text{where} \quad C_i = \frac{\int_0^2 x \sin(z_i x / 2) dx}{\int_0^2 [\sin(z_i x / 2)]^2 dx}.
\]

For large \( y \) we get a valid approximation by keeping only the first term of the series:

\[
\Phi(x,y) \approx C_1 \exp\left(-\frac{z_1 y}{2}\right) \sin\left(\frac{z_1 x}{2}\right), \quad \text{where} \quad C_1 = \frac{\int_0^2 x \sin(z_1 x / 2) dx}{\int_0^2 [\sin(z_1 x / 2)]^2 dx} = 1.375.
\]

(8) Answer already given.

(9) \( T(r,\phi) = \frac{\gamma}{6} (r^2 - a^2) + \frac{T_0}{3} \left[-1 + \frac{4r^2}{a^2} P_2(\cos \phi)\right] \).

(10) Hint: try \( T(r,t) = \frac{\Psi(r,t)}{r} \). Answer: \( T(r,t) = 2T_0 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n \pi^2 D_0}{n \pi} \frac{a}{r} \sin\left(\frac{n \pi r}{a}\right) \).

(11) Lowest angular frequency = \( e \left(\frac{\alpha_1}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2 \) where \( \alpha_1 = 2.40483 \) is the first root of \( J_0 \).

(12) Use the expansion obtained in class for \( a^2 - r^2 \). Then \( \Phi(r,z) = 8 \Phi_0 \sum_{n=1}^{\infty} e^{-\frac{\alpha_n z}{a}} J_0 \left(\frac{\alpha_n r}{a}\right) \),

where \( \alpha_n \) is the \( n \)th positive zero of \( J_0 \).