We look for normal modes of vibration in the form \( y(x,t) = \cos(wt)F(x) \). (We could equally well use \( \sin(wt), \cos(wt) + b\sin(wt), \) or \( a\cos(wt-w) \) for the time part.) We substitute this into the equation to get
\[
\frac{d^4F}{dx^4} = \lambda F, \quad \text{where} \quad \lambda = \frac{w^2}{c^2}.
\]

The boundary conditions require
\[
F(0) = 0, \quad F''(0) = 0, \quad F(L) = 0, \quad F''(L) = 0.
\]

We observe that a sine function is an eigenfunction of \( d^4/dx^4 \). Also if the sine vanishes at a point, so will its second derivative. This suggests we should try \( F(x) = \sin(ax) \). The equation then gives \( \lambda = a^2 \). The boundary conditions for \( x = 0 \) are satisfied for any \( a \). The boundary conditions at \( x = L \) both require \( \sin(aL) = 0 \), hence
\[
\beta_n L = n\pi, \quad \lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2.
\]

The frequency of the mode is
\[
\omega_n = \sqrt{\lambda_n c} = \left(\frac{n\pi}{L}\right) \sqrt{\frac{c}{L}}.
\]

The linear frequency is
\[
\omega_n = \frac{\omega_n}{2\pi} = \frac{\pi}{2} \left(\frac{n\pi}{L}\right) \sqrt{\frac{c}{L}}.
\]

The fundamental frequency is
\[
\omega_1 = \frac{\pi}{2L} \sqrt{\frac{c}{L}}.
\]
We should check for zero and negative eigenvalues. Although given the physical problem the negative eigenvalue are improbable— they would give modes growing in time. We use a Rayleigh Quotient. Multiply the original equation by F. The lefthand side is then

\[ F F^{\prime\prime} = (F F^{\prime\prime}) - F^{\prime}\overset{\prime}{F}^{\prime\prime} = (F F^{\prime\prime} - F^{\prime}\overset{\prime}{F}^{\prime\prime}) + (F^{\prime})^2 \]

Then

\[ (F F^{\prime\prime} - F^{\prime}\overset{\prime}{F}^{\prime\prime}) + (F^{\prime})^2 = \lambda (F)^2 \]

We integrate over \([0, L]\). The first term entirely \( \int_0^L (F F^{\prime\prime} - F^{\prime}\overset{\prime}{F}^{\prime\prime}) dx \) which vanishes because of the boundary conditions. Then

\[ \lambda = \frac{\int_0^L (F^{\prime\prime})^2 dx}{\int_0^L (F^2)^2 dx} \geq 0. \]

If \( \lambda = 0 \), then \( F^{\prime\prime} = 0 \Rightarrow F = A x + B \), but \( F(0) = 0 \)
\( \Rightarrow B = 0 \) and \( F(L) = 0 \Rightarrow A = 0 \), trivial solution.

Here is a more systematic treatment which does not require the slightly inspired guess that \( F(x) = \sin(bx) \).

The projection is

\[ (D^2 - b^2) F = 0, \quad D = \frac{dx}{dx} \]

Then

\[ (D^2 - b^2)(D^2 + b^2) F = 0 \]

or

\[ (D^2 + b^2)(D^2 - b^2) F = 0. \]

We get two families of solutions.

\( (D^2 + b^2) F = 0 \Rightarrow F = A \cos(bx) + B \sin(bx) \)
\( (D^2 - b^2) F = 0 \Rightarrow F = C \cosh(bx) + D \sinh(bx) \)

The general solution is

\[ F = A \cos(bx) + B \sin(bx) + C \cosh(bx) + D \sinh(bx) \]
(1) (c) (continued)

The boundary condition \( F(0) = 0 \) gives \( A + C = 0 \).

The boundary condition \( F'(0) = 0 \) gives \( -A + C = 0 \),
so \( A = 0 \), \( C = 0 \). The boundary condition \( F(L) = 0 \)
gives
\[
B \sin BL + D \sinh BL = 0.
\]
The boundary condition \( F''(L) = 0 \) gives
\[
-BSin BL + D \cosh BL = 0
\]
so \( D \sinh BL = 0 \) and \( B \sin BL = 0 \). The
\( \sinh \) vanishes only at \( B = 0 \), which we already
know is not an eigenvalue, hence \( D = 0 \). \( B \)
(cannot be \( \pm \infty \)), for the solution would then be
trivial.

\( \sin(BL) = 0 \), \( B \neq 0 \),
and the rest is as before.

(b) We are given that \( \nu_2 = \frac{2.33}{\text{Hz}} \). We
have
\[
\nu_2 = \frac{\pi}{2a^2} \sqrt{\frac{E}{\rho}}
\]
so
\[
\sigma = \frac{\pi^2}{a^2} \nu_2^2 = \frac{4 \pi^2 \nu_2^2}{E} \left( \frac{2.33}{\text{Hz}} \right)^2
= 1.20 \frac{\text{kg}}{\text{m}^2 \text{s}^2}.
\]

We have \( \sigma = \frac{E I}{A H} \), so \( \rho = \frac{E I}{\sigma A H} \)

\( E = 2.07 \times 10^9 \text{ N/m}^2 \), \( A = 41 \text{ m}^2 \)
\( \sigma = 1.20 \frac{\text{kg}}{\text{m}^2 \text{s}^2} \), \( \rho = \frac{E I}{\sigma A H} \)
\( I = \frac{W b^3}{12} = \frac{(3 \times 10^{-2} \text{m})^3 (10^{-3} \text{m})^3}{12} = 2.5 \times 10^{-12} \text{ m}^4 \)

\( \text{Then} \quad \rho = \frac{(2.07 \times 10^9) (2.5 \times 10^{-12})}{(2.24 \times 10^{-5})} = 784 \frac{\text{kg}}{\text{m}^2} \)

which is a typical density for steel.

For the second mode, \( \nu_2 = 4 \nu_1 = 9.32 \text{ Hz} \).
(2) (a) The proof is similar in structure to our proof of orthogonality for eigenfunctions of Sturm-Liouville Theory.

Multiply \( \nabla^2 \psi = -\lambda \psi \), by \( \psi_2 \),
multiply \( \nabla^2 \psi_2 = -\lambda_2 \psi_2 \) by \( \psi_1 \), and subtract \( \nabla_2 \nabla^2 \psi_1 = \psi_1 \nabla^2 \psi_2 = (\lambda - \lambda_2) \psi_1 \psi_2 \).

We integrate both terms on the left by parts,

\[
\psi_2 \nabla^2 \psi_1 - \psi_1 \nabla^2 \psi_2 = \psi_2 \nabla \cdot (\nabla \psi_1) - \psi_1 \nabla \cdot (\nabla \psi_2)
\]

\[
- \Delta \cdot (\psi_1 \nabla \psi_2) + (\psi_1 \cdot \Delta \psi_2)
\]

\[
= \nabla \cdot (\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2) = (\lambda_2 - \lambda_1) \psi_1 \psi_2.
\]

Integrate over the volume \( V \) and use the divergence theorem on the left to get

\[
\iint_S [\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2] \cdot \mathbf{n} dS = (\lambda_2 - \lambda_1) \iiint_V \psi_1 \psi_2 dV.
\]

Because \( \psi_1, \psi_2 \) vanish on \( S \) and because \( \lambda_1 \neq \lambda_2 \), we get

\[
\iiint_V \psi_1 \psi_2 dV = 0.
\]

(b) We start with \( \nabla^2 \psi = -\lambda \psi \), and we multiply the equation by \( \psi \). Then the left-hand side is

\[
\psi \nabla^2 \psi = \psi \nabla \cdot (\nabla \psi) = \nabla (\psi \nabla \psi) - (\nabla \psi)^2,
\]

so

\[
\nabla (\psi \nabla \psi) - (\nabla \psi)^2 = -\lambda \psi.
\]

We integrate over \( V \). The first term on the left becomes

\[
\iiint_V \nabla (\psi \nabla \psi) dV = \iiint_V \psi \nabla \cdot (\nabla \psi) dV = 0 \quad \text{by Gauss' law} \lambda_1 < 0.
\]

Then

\[
\lambda = \frac{\iiint_V (\nabla \psi)^2 dV}{\iiint_V \psi^2 dV}.
\]

Hence \( \lambda < 0 \). If \( \psi = 0 \), \( \psi = 0 \), so

\[
\psi = \nabla \psi = 0 \Rightarrow \psi = 0.
\]

So

\[
\psi(\mathbf{r}) = 0 \quad \text{on} \quad \partial V.
\]
The Fourier transform is
\[ \tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ix} (1+x^2) e^{ikx} dx. \]

Because \( f(x) \) is even, we may write this as
\[ \tilde{f}(k) = 2\int_0^{\infty} e^{-x} (1+x^2) \cos(kx) dx. \]

We use Mathematica to get
\[ \tilde{f}(k) = \frac{2(3-4k^2+k^4)}{(1+k^2)^3}. \]

(b) We have \( \tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \), so
\[ \tilde{f}(0) = \int_{-\infty}^{\infty} f(x) dx. \]

From our result for \( \tilde{f}(k) \), we get \( \tilde{f}(0) = 6 \). We have
\[ \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} e^{-ix} (1+x^2) dx = 2\int_{0}^{\infty} e^{-x} (1+x^2) dx, \]
\[ \int_{0}^{\infty} e^{-x} dx = -e^{-x}\bigg|_{0}^{\infty} = 1. \]

We can use Mathematica or integration by parts to evaluate \( \int_{0}^{\infty} e^{-x} x^2 dx \), or we can use the reduction formula. Let \( F(a) = \int_{0}^{\infty} e^{-ax} dx = \frac{1}{a}. \)

Then \( \frac{d^2F}{da^2} = \int_{0}^{\infty} x^2 e^{-ax} dx = \frac{2}{a^2}, \)
so by setting \( a = 1 \), we get \( \int_{0}^{\infty} x^2 e^{-x} dx = 2. \)

Then
\[ \int_{-\infty}^{\infty} f(x) dx = 2 \left\{ \int_{0}^{\infty} e^{-x} dx + \int_{0}^{\infty} x^2 e^{-x} dx \right\} = 2 (1 + 2) = 6 = \tilde{f}(0). \]
(3) (c) We use mathematics to evaluate the integrals.

\[
\int_{-\infty}^{\infty} f(x)^2 dx = 2 \int_{0}^{\infty} e^{-2x} (1 + x^2)^2 dx = \frac{7}{2}
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(k)|^2 dk = \frac{1}{\pi} \int_{0}^{\infty} \left[ \frac{2(3 - 4k^2 + k^4)}{(1 + k^2)^3} \right] dk = \frac{7}{2}
\]

(4) (a) We let \( \tilde{y}(k) \) be the Fourier transform of \( y(x) \). We transform the equation, using the derivative rule for \( y''(x) \):

\[-k^2 \tilde{y} - \tilde{y} = \tilde{f} \cdot e^{-i \pi} (1 + k^2)^2 = \frac{2(3 - 4k^2 + k^4)}{(1 + k^2)^3},\]

Then

\[\tilde{y} = \frac{2(3 - 4k^2 + k^4)}{(1 + k^2)^3}.\]

(b) Then

\[y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{y}(k) dk\]

\[= \frac{1}{\pi} \int_{0}^{\infty} \cos(kx) \cdot \frac{2(3 - 4k^2 + k^4)}{(1 + k^2)^4} dk\]

\[= \frac{e^{-x^2}}{12} \left\{ 9 + \frac{9}{1!} + 3x^2 + 2x^3 \right\} \]

(c) See the Mathematica notebook for a direct check of this solution and for the plot.

**Challenge Problem**

(5) We try \( y = \cos(\omega t) f(x) \). Substituting this into the equation for \( y \), we get

\[
\frac{d^4 F}{dx^4} = \lambda F, \quad 0 < x < 1, \quad \text{with} \quad \lambda = \frac{c \omega^2}{\delta}.
\]

From the BC's only we get \( F(0) = 0, F'(0) = 0, F(1) = 0, F''(1) = 0 \).
(b) We start with
\[ F_1^{(4)} = \lambda_1 F_1, \quad F_2^{(4)} = \lambda_2 F_2. \]
Multiply the first equation by \( F_2 \), the second by \( F_1 \) and then subtract the second from the first:
\[ F_2 F_1^{(4)} - F_1 F_2^{(4)} = (\lambda_1 - \lambda_2) F_1 F_2. \]
We do some integrations by parts on the left-hand side:
\[ F_2 F_1^{(4)} - F_1 F_2^{(4)} = (F_2 F_1^{(3)} - F_1 F_2^{(3)})' = (F_2 F_1^{(2)} - F_1 F_2^{(2)})' \]
Now we integrate over \([0, 2]\) to get
\[ \int_0^2 (F_2 F_1^{(2)} - F_1 F_2^{(2)})' \, dx = (\lambda_1 - \lambda_2) \int_0^2 F_1 F_2 \, dx. \]
From the BC's we find that the left-hand side vanishes. Then for \( \lambda_1 \neq \lambda_2 \) we get
\[ \int_0^2 F_1 F_2 \, dx = 0. \]
(c) We derived a Rayleigh quotient in problem (b) but the boundary conditions are different here, so we need to repeat it. We start by multiplying the equation by \( F_1 \):
\[ F F^{(4)} = \lambda F^2 \]
\[ (F F^{(3)})' - F F^{(2)} = \lambda F^2 \]
\[ (F F^{(2)})' + F F^{(1)} = \lambda F^2. \]
We integrate over \([0, 2]\) to get
\[ \int_0^2 (F F^{(2)})' + \int_0^2 (F F^{(1)})' \, dx = \lambda \int_0^2 F^2 \, dx. \]
It is easy to show from the BC's that the first term on the left is zero. Then
\[ \lambda = \frac{\int_0^2 (F F^{(1)})' \, dx}{\int_0^2 (F F^{(2)})' \, dx}. \]
The Rayleigh quotient shows that $\lambda \geq 0$. If $\lambda = 0$, $F''$ must vanish, hence $F = A x + B$. $F(0) = 0 \Rightarrow B = 0$
and $F'(0) = 0 \Rightarrow A = 0$, so $\lambda = 0$ gives only the trivial solution.

(c) We let $\lambda = \beta^2$. Then
\[
\frac{d^4F}{dx^4} - \beta^4 F = 0
\]
or $(D^4 - \beta^4) F = 0$
or $(D^2 - \beta^2)(D^2 + \beta^2) F = 0$
or $(D^3 + \beta^2)(D^2 - \beta^2) F = 0$.
From $(D^2 + \beta^2) F = 0$ we get solutions $\cos \beta x$ and $\sin \beta x$. From $(D^3 + \beta^2)(D^2 - \beta^2) F = 0$ we get solutions $\cos \beta x$ and $\sin \beta x$. The general solution is then
\[
F(x) = A \cos \beta x + B \sin \beta x + C \cos \beta x + D \sin \beta x.
\]
(e) We have $F(0) = 0$ which gives $A + C = 0$.
We have $F'(0) = 0$ which gives $B + D = 0$.
We eliminate $C$ and $D$ to get
\[
F(x) = A (\cos \beta x - \cos \beta x) + B (\sin \beta x - \sin \beta x),
\]
Now we impose the BC's at $x = L$.
$F(L) = 0 \Rightarrow A (\cos \beta L - \cos \beta L) + B (\sin \beta L - \sin \beta L)$
$F'(L) = 0 \Rightarrow A (\cos \beta L + \cos \beta L) + B (\sin \beta L + \sin \beta L)$
We now have a set of two simultaneous linear homogeneous algebraic equations. The condition for a non-trivial solution is that the determinant of the coefficient matrix should vanish. Hence
\[
0 = (\cos \beta L - \cos \beta L)(\sin \beta L + \sin \beta L) - (\sin \beta L - \sin \beta L)(\cos \beta L + \cos \beta L)
\]
\[= 2 \cos \beta L \sin \beta L - 2 \sin \beta L \cos \beta L
\]
We divide by $2 \cos \beta L \cos \beta L$ to get
\[
\tan (\beta L) = \frac{\cos \beta L}{\sin \beta L}
\]
Challengers Problem (Continued)

(f) From Mathematica we get \( z_1 = 3.9266 \), \( z_2 = 7.0686 \), and \( z_3 = 10.2102 \). We also show in the Mathematica notebook that a good approximation to the \( n \)th root \( z_n \) is 
\[ z_n \approx (n + \frac{1}{2})\pi. \]

(g) We have 
\[ \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{200 \text{ m}/\text{s}^2}} = \frac{2}{200 \text{ m}/\text{s}^2}. \]

and

\[ \nu_1 = \frac{\omega_1}{2\pi} = \frac{2}{200 \text{ m}/\text{s}^2} \sqrt{\frac{2.200 \text{ m}^2}{52}}. \]

In Problem (i), we calculated \( \sigma = 2.200 \text{ m}/\text{s}^2 \). So

\[ \nu_1 = \frac{(3.9266)^2}{2\pi (1\text{ m})^2} \sqrt{\frac{2.200 \text{ m}^2}{52}} = 3.6397 \text{ Hz}. \]

This is higher than the frequency calculated in problem (i). The clamp support on the left gives additional stiffness.
ME201/MTH281/ME400/CHE400
Assignment #8 Solutions
Problems 3 and 4, and the Challenge Problem

Problem 3

- (a)
  We use Mathematica to calculate the Fourier transform.

\[
\text{ftran}[k_] = 2 \text{Simplify}\left[ \text{Integrate}\left[ \exp[-x] \left( 1 + x^2 \right) \cos(kx), \{x, 0, \infty\}\right], \{k, 0, \infty\}\right], \text{Reals}\]
\[
\frac{2 \left( 3 - 4 k^2 + k^4 \right)}{(1 + k^2)^3}
\]

- (c)

\[
\text{Integrate}\left[ \exp[-2 x] \left( 1 + x^2 \right)^{\frac{1}{2}}, \{x, 0, \infty\}\right]
\]
\[
\frac{7}{2}
\]
\[
\text{(1/Pi) Integrate}\left[ \text{ftran}[k]^2, \{k, 0, \infty\}\right]
\]
\[
7\]
\[
\frac{1}{2}
\]

Problem 4

- (b)
  We use Mathematica to evaluate the integral arising in the inverse Fourier Transform.

\[
y[x_] = \frac{1}{\pi} \text{Integrate}\left[ \cos(kx) \text{ftran}[k] / \left( k^2 + 1 \right), \{k, 0, \infty\}, \text{Assumptions} \rightarrow k \in \text{Reals}\right]
\]
\[
\frac{1}{12} e^{-\text{Abs}[x]} \left( 9 + 3 x^2 + 9 \text{Abs}[x] + 2 \text{Abs}[x]^3 \right)
\]

We see that this is an even function of x. We use this to simplify the formula for positive and negative x. We let ypos[x] be the solution when x>=0, and yneg[x] be the solution when x<0.

\[
ypos[x_] = \text{Simplify}[y[x], x > 0]
\]
\[
\frac{1}{12} e^{-x} \left( 9 + 9 x + 3 x^2 + 2 x^3 \right)
\]
$y_{neg}[x_] = \text{Simplify}[y[x], x < 0]$

\[ \frac{1}{12} e^x \left( 9 - 9 x + 3 x^2 - 2 x^3 \right) \]

Now we check that the differential equation is satisfied. First for $x > 0$.

\[ \text{Simplify}[D[y_{pos}[x], \{x, 2\}] - y_{pos}[x] + \text{Exp}[-x] \{1 + x^2\}] \]

Out[17] = 0

Now for $x < 0$.

\[ \text{Simplify}[D[y_{neg}[x], \{x, 2\}] - y_{neg}[x] + \text{Exp}[x] \{1 + x^2\}] \]

Out[18] = 0

Now we check for smoothness at the origin.

\[ y_{pos}[0] \]

Out[19] = 3

\[ y_{neg}[0] \]

Out[20] = 3

\[ D[y_{pos}[x], x] / . x \to 0 \]

Out[21] = 0

\[ D[y_{neg}[x], x] / . x \to 0 \]

Out[22] = 0

\[ D[y_{pos}[x], \{x, 2\}] / . x \to 0 \]

Out[23] = -\frac{1}{4}

\[ D[y_{neg}[x], \{x, 2\}] / . x \to 0 \]

Out[24] = -\frac{1}{4}

\[ D[y_{pos}[x], \{x, 3\}] / . x \to 0 \]

Out[25] = 1

\[ D[y_{neg}[x], \{x, 3\}] / . x \to 0 \]

Out[26] = -1

We see that the function, and its first and second derivatives are continuous at the origin, but the third derivative is not. The origin of this discontinuity is in the source function.

\[ (c) \]

Now we plot the solution.
Challenge Problem

(f)

We begin by plotting the two sides of the eigenvalue equation.

In[11] := \text{Plot}\{\{\text{Tan}[z], \text{Tanh}[z]\}, \{z, 0, 12\}, \text{AxesLabel} \rightarrow \{"z", "\text{Tan}[z] \text{and Tanh}[z]"\}\}

Out[11] =

We see that there are roots near 4, 7, and 10. We use \text{FindRoot} to get accurate values.

In[12] := z1 = z /. \text{FindRoot}[\text{Tan}[z] = \text{Tanh}[z], \{z, 4\}]

Out[12] = 3.9266

In[13] := z2 = z /. \text{FindRoot}[\text{Tan}[z] = \text{Tanh}[z], \{z, 7\}]

Out[13] = 7.06858

In[14] := z3 = z /. \text{FindRoot}[\text{Tan}[z] = \text{Tanh}[z], \{z, 10\}]

Out[14] = 10.2102

We notice that the function crossings occur at places where the tanh is very nearly 1. Thus an approximate eigenvalue equation is \text{Tan}[z] = 1, which gives \( z = \pi / 4 \) \pm any multiple of \( \pi \). This suggests for \( z \) the approximation

\text{approx}[z] := \pi / 4 \pm n \pi

We try this for the first three eigenvalues.
The approximation is both simple and accurate.