(1) (a) The solution must vanish at \(x = 0\) and \(x = a\), so we look for separated solutions which are oscillatory in \(x\). We try \(\Phi = F(x)G(y)\). We substitute into the equation to get

\[
\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.
\]

By the usual separation arguments, both sides are equal to the same constant. For \(F(x)\) to be oscillatory, we want \(F''(x) < 0\), so we take the constant to be \(-\lambda\). Then

\[
F''(x) + \lambda F(x) = 0, \quad 0 < x < a
\]

with \(F(0) = 0, F(a) = 0\). This is a familiar problem and the result is

\[
\lambda = \lambda_n = \frac{n^2 \pi^2}{a^2}, \quad F_n(x) = \sin \left(\frac{n \pi x}{a}\right), \quad n = 1, 2, 3, \ldots
\]

The associated \(y\)-part is \(G_n(y)\), when

\[
\frac{G''(y)}{G(y)} = -\frac{n^2 \pi^2}{a^2} G_n(y) = 0
\]

We have a choice of solution basis here. If we make the choice carefully, we can save a lot of labor, although we will get the same final answer no matter which basis we choose. One possible basis is

\[
\begin{bmatrix}
  e^{\alpha y} \\
  e^{-\alpha y}
\end{bmatrix}
\]

This basis would be useful in an unbounded geometry because conditions \(\alpha = 0\) would eliminate headache exponential) is unbounded. For bounded geometry, the hyperbolic
This will work, but there is an even better choice than this, and that is
\[
\left\{ \sinh \left( \frac{ny}{a} \right), \sinh \left( \frac{ny(b-y)}{a} \right) \right\}.
\]

For this basis, the first function vanishes at \( y = 0 \) and the second vanishes at \( y = b \).

This will give us a relatively easy way to calculate the coefficients from the boundary conditions. Our separated solutions are then
\[
\Phi = \sum_{n=0}^{\infty} \left[ A_n \sinh \left( \frac{ny}{a} \right) + B_n \sinh \left( \frac{ny(b-y)}{a} \right) \right] \sin \left( \frac{n\pi x}{a} \right).
\]

We try
\[
\Phi = \sum_{n=0}^{\infty} \left[ A_n \sinh \left( \frac{ny}{a} \right) + B_n \sinh \left( \frac{ny(b-y)}{a} \right) \right] \sin \left( \frac{n\pi x}{a} \right).
\]

We do the easy boundary condition at \( y = b \)
first:
\[
\Phi(x, b) = \Phi_0 \sin \left( \frac{n\pi x}{a} \right) = \sum_{n=0}^{\infty} A_n \sinh \left( \frac{ny}{a} \right) \sin \left( \frac{n\pi x}{a} \right).
\]

We balance coefficients to get
\[
A_n = \frac{\Phi_1}{\sinh \left( \frac{ny}{a} \right)}, \quad A_n = 0 \text{ for } n > 1.
\]

Now we impose the boundary condition at \( y = 0 \):
\[
\Phi(x, 0) = \Phi_0 \frac{x}{a} (1 - \frac{x}{a}) = \sum_{n=0}^{\infty} B_n \sinh \left( \frac{ny}{a} \right) \sin \left( \frac{n\pi x}{a} \right).
\]
(1) (a) (continued). This is a Fourier sine series on the interval \([0, a]\), so we get

\[
B_n = \frac{2}{a} \frac{1}{\sinh \left( \frac{n \pi b}{a} \right)} \int_0^a x \sin \left( \frac{n \pi x}{a} \right) dx.
\]

We use Mathematica to evaluate the integral. We get

\[
B_n = 0 \quad \text{for even} \ n,
\]

and \(B_n = \frac{8 \Phi_0}{n^3 \pi^3} \frac{1}{\sinh \left( \frac{n \pi b}{a} \right)} \quad \text{for odd} \ n.
\]

Then

\[
\Phi(x, y) = \Phi_1 \frac{\sinh \left( \frac{n \pi y}{a} \right)}{\sinh \left( \frac{n \pi b}{a} \right)} \sin \left( \frac{n \pi x}{a} \right)
\]

\[
+ \frac{8 \Phi_0}{n^3 \pi^3} \sum_{n=1, \text{odd}}^{\infty} \frac{1}{n^3} \frac{\sin \left( \frac{n \pi (b-y)}{a} \right)}{\sinh \left( \frac{n \pi b}{a} \right)} \sin \left( \frac{n \pi x}{a} \right)
\]

If we had chosen the \(y\)-basis \(\frac{\sinh \left( \frac{n \pi y}{a} \right)}{\sinh \left( \frac{n \pi b}{a} \right)}\)

\(\cos \left( \frac{n \pi y}{a} \right)\)

we would have gotten a more complicated expression which could be put into the above form by using the addition formulas for the hyperbolic functions.

(b) Set Mathematica notebook.

(2) Because \(\Phi\) vanishes at \(x = 0\) and \(x = a\), we expect the solutions to be oscillatory in \(x\). We try \(\Phi = F(x)G(y)\). The equation gives
(2) (continued)

\[ G(y)F''(y) + F(x)G''(y) = 0 \]

We divide by \( FG \) to get

\[ \frac{F''}{F} = -\frac{G''}{G} \]

The separation has worked. Both sides of the equation must be equal to the same constant which we denote by \( -\lambda \). Then the \( x \)-problem is

\[ F'' + \lambda F = 0, \quad 0 < x < a \]

\[ F(0) = 0, \quad F(a) = 0 \]

This is a problem we have solved several times. The result is

\[ \lambda = \lambda_n = \frac{n^2 \pi^2}{a^2}, \quad F_n(x) = \sin \left( \frac{n \pi x}{a} \right) \]

\[ n = 1, 2, \ldots \]

The associated \( y \)-problem is

\[ -\frac{G''}{G} + \delta = -\lambda_n \]

so \( G'' - (\delta + \lambda_n) G_n = 0 \).

We also require that \( G_n \to 0 \) as \( y \to \infty \).

The inhomogeneous boundary condition cannot be imposed until we have constructed a superposition of all the solutions.

We set \( \delta_n = \sqrt{\delta + \lambda_n} \). Then the general solution for \( G_n \) is

\[ G_n = e^{-\delta y} + B_n e^{\delta y} \]

The positive exponential blows up as \( y \to \infty \) so we must have \( B_n = 0 \). Then \( G_n \approx e^{-\delta y} \).

Our separated solutions are

\[ \Phi_n(x, y) = e^{-\delta y} \sin \left( \frac{n \pi x}{a} \right), \quad \delta = \sqrt{\delta + \lambda_n} \]
(2) (continued) We superpose our solutions to get Φ.

\[ \Phi(x,y) = \sum_{n=1}^{\infty} C_n e^{-\theta_n y} \sin \left( \frac{n\pi x}{a} \right). \]

Provided this series converges, it satisfies both the equation and the homogeneous boundary conditions at \( x = 0, a \). The remaining condition is the inhomogeneous boundary condition at \( y = 0 \).

\[ \Phi(x,0) = \Phi_0 = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi x}{a} \right) \]

This is a Fourier sine series, and we (circulate \( C_n \) in the usual way by orthogonality:

\[ C_n = \frac{2}{a} \int_0^a \Phi_0 \sin \left( \frac{n\pi x}{a} \right) \, dx \]

\[ = \frac{2 \Phi_0}{a} \int_0^a \sin \left( \frac{n\pi x}{a} \right) \, dx \]

\[ = 0 \quad \text{for} \quad n \, \text{odd} \]

\[ \frac{\Phi_0}{\sin \left( \frac{n\pi}{a} \right)} \quad \text{for} \quad n \, \text{even} \]

Our solution is then

\[ \Phi(x,y) = \frac{\Phi_0}{\pi} \sum_{\substack{n=1 \, \text{odd} \n}}^{\infty} e^{-\theta_n y} \sin \left( \frac{n\pi x}{a} \right) \]

where \( \theta_n = \sqrt{\theta_0^2 + \frac{n^2\pi^2}{a^2}} \).

(3) (a) We have \( \frac{\partial}{\partial t} \frac{1}{2} \left( \frac{\partial y}{\partial t} \right)^2 = \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \) and

\( \frac{\partial}{\partial t} \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2 = \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} \), so
\[ \frac{dE}{dt} = 6 \int_0^L \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} \, dx + 4 \int_0^L \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} \, dx. \]

From this partial differential equation we have
\[ 0 \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - \epsilon \frac{\partial y}{\partial t}. \]

(3)
We substitute this into the expression for \( \frac{dE}{dt} \) to get
\[ \frac{dE}{dt} = 7 \int_0^L \left( \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} \right) \, dx \]
\[ \quad - \epsilon \int_0^L \left( \frac{\partial y}{\partial t} \right)^2 \, dx. \]

For the integrand in the first term we have
\[ \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial x \partial t} = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right), \]
so the first terms is
\[ T \int_0^L \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right) \, dx = T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \bigg|_0^L. \]

At \( x=0 \) and \( L \), \( y=0 \) for \( \partial y/\partial x \), so \( \partial y/\partial x \) vanishes at \( x=0 \) and \( L \). Thus this term is zero. Then
\[ \frac{dE}{dt} = -\epsilon \int_0^L \left( \frac{\partial y}{\partial t} \right)^2 \, dx. \]

(c) The energy will decrease as long as the string is in motion.
(5) (c) We have \( P \frac{\partial T_1}{\partial t} = \nabla \cdot (KV T_1) + \Gamma(T1) \)

and \( P \frac{\partial T_2}{\partial t} = \nabla \cdot (KV T_2) + \Gamma \).

We subtract the second equation from the first and use the linearity of \( \frac{\partial}{\partial t} \) and \( \nabla \) to get

\[ P \frac{\partial (T_1 - T_2)}{\partial t} = \nabla \cdot (KV(T_1 - T_2)) \]

or \( P \frac{\partial \hat{T}}{\partial t} = \nabla \cdot (KV \hat{T}) \).

The boundary condition: \( \hat{T}|_S = T_1|_S - T_2|_S = f - f = 0 \).

The initial condition: \( \hat{T}(x,0) = T_1(x,0) - T_2(x,0) = g - g = 0 \).

The problem for \( \hat{T} \) is

**Equation:** \( P \frac{\partial \hat{T}}{\partial t} = \nabla \cdot (KV \hat{T}) \) \( \text{in } V \)

**BC:** \( \hat{T}|_S = 0 \)

**IC:** \( \hat{T}(x,0) = 0 \).

(b) Let \( E = \iiint \frac{1}{2} P \hat{T}^2 \, dx \).

Then \( \frac{\partial E}{\partial t} = \iiint P \hat{T} \cdot \frac{\partial \hat{T}}{\partial t} \, dx \).

From the partial differential equation, we have \( P \frac{\partial \hat{T}}{\partial t} = \nabla \cdot (KV \hat{T}) \).

We substitute this into \( \frac{\partial E}{\partial t} \):
\[ \frac{dE}{dt} = \iiint \nabla \cdot (k \nabla T) \, dV. \]

We do an integration by parts.

\[ \nabla \cdot (k \nabla T) = \nabla \cdot (T \nabla k) - \nabla k \cdot \nabla T = \nabla \cdot \left( T \nabla k \right) - k \left( \nabla T \right)^2. \]

Then

\[ \frac{dE}{dt} = \iiint \nabla \cdot \left( T \nabla k \right) \, dV - \iiint k \left( \nabla T \right)^2 \, dV. \]

We use the divergence theorem on the first term to get

\[ \iiint \nabla \cdot \left( T \nabla k \right) \, dV = \int_S T \nabla k \cdot \nu \, dS. \]

On \( S \), \( T = 0 \) so this last integral is zero.

Then

\[ \frac{dE}{dt} = -\iiint k \left( \nabla T \right)^2 \, dV \leq 0. \]

(c) Because \( E = \frac{1}{2} \int_0^1 \left( T^2 \right) \, dx \) is the integral of a product of functions which are never negative, \( E \geq 0 \). Because \( T(1,0) = 0 \)

\[ E(1,0) = 0. \]

This, in combination with

\[ \frac{dE}{dt} \leq 0 \]

shows that \( E(t) \leq 0 \), hence \( E(t) = 0 \) for all \( t \), which can happen only if \( T = 0 \) for all \( t \). Hence \( T = T_2 \) is unique.
Suppose there are two solutions \( T_1 \) and \( T_2 \). Let \( \tilde{T} = T_1 - T_2 \). Then

\[
\nabla^2 \tilde{T}_1 = \tilde{T} \text{ in } V
\]

and \( \nabla^2 T_2 = \tilde{T} \text{ in } V \).

By subtracting the equations we find that

\[
\nabla^2 \tilde{T} = 0 \text{ in } V.
\]

The boundary condition is

\[
\eta \cdot \nabla \tilde{T} \big|_S = \eta \cdot \nabla T_1 \big|_S - \eta \cdot \nabla T_2 \big|_S = f - f = 0.
\]

Multiply the equation for \( \tilde{T} \) by \( \tilde{T} \) and integrate by parts.

\[
0 = \int \nabla \tilde{T} \cdot \nabla \tilde{T} - \int (\nabla \tilde{T})^2
\]

Integrate this over \( V \),

\[
0 = \iiint \nabla \cdot (\tilde{T} \nabla \tilde{T}) \, dx - \iiint (\nabla \tilde{T})^2 \, dx
\]

Use the divergence theorem on the first term to get

\[
\int \tilde{T} \nabla \cdot \eta \, dx = 0.
\]

But \( \eta \cdot \nabla T_2 \big|_S = 0 \), so this term vanishes. This gives

\[
\iiint (\nabla \tilde{T})^2 \, dx = 0
\]

\[
\Rightarrow \nabla \tilde{T} = 0 \Rightarrow \tilde{T} = \text{ constant}.
\]

Thus \( T = T_2 + \text{ constant} \).
If we use the general notation of problem (1), we have
\[ \nabla^2 T = \frac{1}{\alpha^2} \nabla^2 V. \]
We integrate this over \( V \).

\[ \iiint_V 2p = \iiint_V \frac{1}{\alpha^2} \nabla^2 V \, dV. \]

So \[ \iiint_S \nabla \cdot \mathbf{u} \, dS = \iiint_V \frac{1}{\alpha^2} \nabla^2 V \, dV. \]

If \( \int = 0 \) as in this problem, we have
\[ \iiint_S \nabla \cdot \mathbf{u} \, dS = 0. \]

We may state the issue here in terms of heat conduction. In steady-state, the total rate of heat addition must equal the total rate of outflow. Otherwise the temperature will change with time.

In this problem,
\[ \iiint_S \nabla \cdot \mathbf{u} \, dS = \int_0^a \int_0^b \frac{\partial T}{\partial t} (x, y) \, dx \, dy \]
\[ = \int_0^a \int_0^b \frac{\partial T}{\partial t} (x, y) \, dx \, dy = \frac{\partial T}{\partial t} \alpha \neq 0. \]

\[ \therefore \text{ no solution.} \]
To say a solution is unique means that there is no more than one solution. This is not inconsistent with there being no solution. Existence and uniqueness are independent concepts and each requires a proof. As we showed in (2), a necessary condition for the existence of a solution is

\[ \iint_{D} \varphi = \iiint_{V} n \cdot \mathbf{F} \, d\sigma. \]

(2) **Additional Comments**

The easiest way to show that there is no solution is the way given previously—show that the condition

\[ \iint_{D} \varphi = \iiint_{V} f \, d\sigma \]

is violated in this case.

It is also instructive to try to solve the problem by separation of variables to see where the solution breaks down. We do that here.

Eq \[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \]

BC \[ \frac{\partial T}{\partial x} (0, y) = 0, \quad \frac{\partial T}{\partial x} (a, y) = 0, \quad \frac{\partial T}{\partial y} (x, 0) = 0 \]

and \[ \frac{\partial T}{\partial y} (x, b) = T_0 \left( \frac{x}{a} \right) \left( 1 - \frac{x}{a} \right). \]

We see that the solutions must be oscillatory in \( x \), so we try

\[ T = F(x)G(y), \]

where we expect \( F(x) \) to be oscillatory.
We get

\[ F'' + \lambda F = 0, \quad 0 < x < a \]

\[ F'(0) = 0, \quad F'(a) = 0 \]

And

\[ G'' - \lambda G = 0, \quad 0 < y < b \]

\[ G'(0) = 0 \]

The x-eqution with zero derivative BC's is

one we have solved before, we get

\[ \lambda_n = \frac{n^2 \pi^2}{a^2}, \quad n = 1, 2, 3, \ldots \]

Now we solve the y-eqution.

\[ G_{0,0}'' = 0, \quad G_{0,0} = G_0 + B_0 \]

\[ G_{0,0}'(0) = 0 \Rightarrow A_0 = 0, \quad \text{so } G_0 = \text{constant} \]

which we take to be 1.

\[ G_n'' - \frac{n^2 \pi^2}{a^2} G_n = 0 \Rightarrow G_n = A_n \cosh \left( \frac{n \pi y}{a} \right) \]

\[ + B_n \sinh \left( \frac{n \pi y}{a} \right) \]

The homogeneous BC gives \( G_n(0) = 0 \Rightarrow \frac{B_n}{a} = 0 \)

so \( B_n = 0 \). Then

\[ G_n(y) = \cosh \left( \frac{n \pi y}{a} \right) \]

Our separate solutions are

\[ T_0(x, y) = 1 \quad T_n(x, y) = \cosh \left( \frac{n \pi y}{a} \right) \cos \left( \frac{n \pi x}{a} \right) \]

Then

\[ T(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh \left( \frac{n \pi y}{a} \right) \cos \left( \frac{n \pi x}{a} \right) \]

This satisfies the equation and the three homogeneous BC's. We impose the inhomogeneous BC:

\[ \frac{\partial T}{\partial y}(x, y) = \frac{T_0(x)}{G(1-x)} = \sum_{n=1}^{\infty} \frac{A_n}{a} \sinh \left( \frac{n \pi y}{a} \right) \cos \left( \frac{n \pi x}{a} \right) \]
(2) ADDITIONAL COMMENTS (continued)

This appears to be a Fourier cosine series, so we get

\[ a_n = \frac{2}{a} \frac{a}{\pi n} \sinh \left( \frac{m \pi b}{a} \right) \int_0^b \frac{x}{a} (1 - \frac{x}{a}) \cos \left( \frac{m \pi x}{a} \right) dx. \]

These coefficients can be calculated easily, and it appears as though we have found a solution. We have not, but the error is subtle. The function \( \frac{x}{a} (1 - \frac{x}{a}) \) has a non-zero mean:

\[ \frac{1}{a} \int_0^b \frac{x}{a} (1 - \frac{x}{a}) dx > 0, \]

but there is no constant term in the series on the right to balance it. If we use the \( a_n \) values above, we will find that our series does not satisfy the boundary condition. If we add a term \( Cy \) to our solution, we would get the constant \( C \) we need in the upper boundary series, but our solution would no longer satisfy the lower boundary condition.
Problem 1

(a) We evaluate the integral appearing in the expression for the coefficient $B_n$.

```mathematica
In[1]:= Simplify[Integrate[Sin[(n \pi x) / a] (x / a) (1 - x / a), {x, 0, a}], n \in Integers]
```

```
Out[1]= -\frac{2 (-1 + (-1)^n) a}{n^3 \pi^3}
```

We see that the coefficient vanishes for even $n$ and for odd $n$ is $(4a)/(n\pi)^3$.

(b) We split the solution into two parts. The part $\Phi_{\text{upper}}$ is the part coming from the upper boundary and is proportional to the constant $\Phi_1$. We define this for Mathematica.

```mathematica
In[2]:= \$upper[x_, y_] := 1 \* \frac{\text{Sinh}[\pi y / a]}{\text{Sinh}[\pi b / a]} \* \text{Sin}[\pi x / a]
```

The other part, which we call $\Phi_{\text{lower}}$ comes from the lower boundary condition and is proportional to the constant $\Phi_0$. The main question with that part is how many terms to keep in the series. The argument of the sinh function in the numerator is less than or equal to the argument of the sinh in the denominator, so the sinh ratio is never larger than 1. (This is conservative, since the ratio away from the upper boundary $y = b$ will be considerably smaller than 1 for large $n$.) The term $1/n^3$ will be less than 0.001 when $n$ exceeds 10, so we may take the sum over $n = 1, 3, 5, 7$ and 9 with an error that we expect to be less than 0.1%. We first define the $n$th term for Mathematica, and then the expression for $\Phi_{\text{lower}}$.

```mathematica
In[3]:= \$term[x_, y_, n_] := 1 \frac{\text{Sin}[\pi (n b - y)] / a}{\text{Sin}[\pi b / a]} \text{Sin}[n \pi x / a]
```

```mathematica
In[4]:= \$lower[x_, y_] := 8 \pi \text{Sum[term[x, y, n], {n, 1, 9, 2}]}
```

In the Sum command, we have asked for the sum from $n = 1$ to $n = 9$ with steps of 2 in $n$. Thus we get only the odd terms as required.

Now we define the solution $\Phi$.

```mathematica
In[5]:= \$[x_, y_] := \$upper[x, y] + \$lower[x, y]
```

Finally we specify the numerical values we will use for the constants.

```mathematica
In[6]:= a = 1.0; b = 1.0; \$0 = 40.0; \$1 = 15.0;
```

We now define a function which produces a graph of $\Phi$ versus $x$ for a given $y$ value.

```mathematica
In[7]:= graph[y_] := Plot[$[x, y], {x, 0, a}, AxesLabel -> {"x (m)" , "\$ (volts)"}, PlotLabel -> Row["y = ", PaddedForm[y, (3, 2)], " (m)"], PlotRange -> (0, 15)]
```
Here are a few comments on the command to produce the graph. The basic plotting command is \texttt{Plot}. The first argument is the function to be plotted. The second argument is a list, specifying the quantity varied in the plot (x in this case), and the initial and final values. The option \texttt{AxesLabel} points to a list containing the labels for the x and y axes. The option \texttt{PlotLabel} constructs an overall label at the top of the graph. The \texttt{Row} functions concatenates various pieces of the desired plot label. Items in quotes are treated as strings. The middle item in the \texttt{Row} list is the value of y, with a number formatting command applied. \texttt{PaddedForm[y,{3,2}]} produces the numerical value of y with 3 digits total and 2 digits to the right of the decimal point. Finally \texttt{PlotRange} specifies the y-range in the plot.

We use this to construct the four graphs requested.

\begin{verbatim}
In[8]:= Do[Print[graph[(n b) / 4], {n, 0, 4}]
    ,
    
y = 0.00 (m)
    
\textbf{y = 0.25 (m)}
\end{verbatim}
The above is all that was required for this part of the problem. However it is of interest to construct a few more graphs.

As a check on our work, we can compare the graphs at $y = 0$ and $b$ to the prescribed boundary values. We do this for $y = 0$ first. We plot the series solution in blue and the given boundary condition in red.
The two curves essentially coincide. We see the red one because it is plotted last.

Now we look at the upper boundary.

The same good result.

The four graphs above give some idea of how \( \Phi \) varies throughout the rectangle. However, we can get a much better idea of this if we construct a contour plot, something which is easy to do because of Mathematica’s built-in contour plot command.
This gives a much better picture of how the potential varies inside the region. Because we used the option ContourLabels -> True, *Mathematica* has put the voltage values on the contours. If we do not use the option ContourShading -> None, we get a nicer looking plot, but one on which it is slightly harder to read the contour values:

\[
\text{In[12]:=} \quad \text{ContourPlot}[\&[x, y], \{x, 0, a\}, \{y, 0, b\}, \text{AspectRatio} \to b/a, \text{ContourLabels} \to \text{True}, \text{ContourShading} \to \text{None}]
\]