(a) Inside a sphere, the relevant separated solutions are $r^n P_n(\cos \phi)$. We try

$$\Phi(r, \phi) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos \phi).$$

If the BC is $\Phi(0, \phi) = g(\phi)$, we may expand in Legendre polynomials:

$$g(\phi) = \sum_{n=0}^{\infty} G_n P_n(\cos \phi).$$

We impose the BC on $\Phi$:

$$\Phi(r, \phi) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos \phi) = g(\phi) = \sum_{n=0}^{\infty} G_n P_n(\cos \phi).$$

We balance coefficients of $P_n$ to get $C_n G_n = G_n$ so

$$\Phi(r, \phi) = \sum_{n=0}^{\infty} C_n \left( \frac{r}{a} \right)^n P_n(\cos \phi).$$

In the present problem, it is straightforward to determine the coefficients $G_n$ without doing any integrations. We have

$$g(\phi) = \Phi_0 \left( \sin^2 \phi \right) = \Phi_0 \left( 1 - \cos^2 \phi \right)$$

Also, $P_2(\cos \phi) = \frac{1}{2} \left( 3 \cos^2 \phi - 1 \right)$, so $\cos^2 \phi = \frac{2P_2 + 1}{3}$.

Then

$$g(\phi) = \Phi_0 \left( 1 - \frac{2P_2 + 1}{3} \right) = \Phi_0 \left( \frac{2}{3} - \frac{2P_2}{3} \right)$$

$$= \Phi_0 \cdot \frac{2}{3} \left( P_0 - P_2 \right),$$

so $G_0 = \frac{2}{3} \Phi_0$, $G_2 = -\frac{2}{3} \Phi_0$ and all other $G_n = 0$.

Then

$$\Phi(r, \phi) = \sum_{n=0}^{\infty} \frac{2}{3} \Phi_0 \left( P_0 - P_2 \right) P_n(\cos \phi)$$

(b) See Mathematica notebook.
(2) (a) The solutions outside a sphere which are \( \Phi(0, \Theta) \) are \( P_n(\cos \Theta) / r^{n+1} \). The solution for \( \Phi \) is obtained by superposition:

\[
\Phi(r, \Theta) = \sum_{n=0}^{\infty} C_n \frac{P_n(\cos \Theta)}{r^{n+1}}.
\]

The boundary condition is \( \Phi(0, \Theta) = g(\Theta) \). We expand \( g \) in Legendre polynomials:

\[
g(\Theta) = \sum_{n=0}^{\infty} G_n P_n(\cos \Theta).
\]

We impose the BC on \( \Phi \):

\[
\begin{align*}
\Phi(0, \Theta) &= \sum_{n=0}^{\infty} C_n \frac{P_n(\cos \Theta)}{r^{n+1}} = g(\Theta) = \sum_{n=0}^{\infty} G_n P_n(\cos \Theta),
\end{align*}
\]

We balance coefficients to get

\[
C_n \frac{1}{r^{n+1}} = G_n, \quad \text{so} \quad C_n = G_n r^{n+1}, \quad \text{and}
\]

\[
\Phi(r, \Theta) = \sum_{n=0}^{\infty} G_n \left( \frac{r}{r} \right)^{n+1} P_n(\cos \Theta).
\]

To finish the problem, we need to calculate the \( G_n \)’s.

We were given \( g(\Theta) = \Phi_0 (1 + \Theta \cos \Theta + \Theta^2 \cos^2 \Theta) \). From the handout on Legendre polynomials we get

\[
\cos^2 \Theta = \frac{1}{2} (P_0 + 2P_2), \quad \cos \Theta = \frac{1}{35} (7P_0 + 20P_2 + 8P_4).
\]

Then \( g(\Theta) = \Phi_0 \left\{ P_0 + \frac{1}{35} (7P_0 + 20P_2 + 8P_4) \right\} \)

\[
= \Phi_0 \left\{ (1 + \frac{1}{35} \Theta^2)P_0 + \left( \frac{2}{3} \Theta + \frac{1}{7} \right)P_2 + \frac{8}{35} P_4 \right\}
\]

The solution is \( \Phi(r, \Theta) = \Phi_0 \left\{ (1 + \frac{1}{35} \Theta^2) \frac{P_0(\cos \Theta)}{r} + \left( \frac{2}{3} \Theta + \frac{1}{7} \right) \frac{P_2(\cos \Theta)}{r^2} + \frac{8}{35} \frac{P_4(\cos \Theta)}{r^4} \right\} \).
(2) (continued)

If we can choose $\alpha$ and $\beta$ so that the first two coefficients vanish, then the drop-off will be like $1/r^5$ and this is the best we can do. So we require

$$1 + \frac{\alpha}{3} + \frac{\beta}{5} = 0, \quad 2 + \frac{2\alpha}{3} + \frac{\beta}{7} = 0.$$ 

We get $\alpha = -10, \beta = \frac{35}{3}$, and

$$\Phi(r, \phi) = \frac{\theta}{3} \phi_0 \frac{P_2(\cos \phi)}{r^5}.$$ 

(3) For a region between two concentric spheres, all of the separated solutions $r^n P_n(\cos \phi)$ and $P_n(\cos \phi)/r^n$ are relevant. We use superposition:

$$\Phi(r, \phi) = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \phi).$$

We look at the general case first before looking at the particular boundary conditions given for this problem.

Let $f(\phi) = \sum_{n=0}^{\infty} a_n P_n(\cos \phi)$ be the boundary condition given on $r = a_0$. Then

$$\Phi(a_0, \phi) = \sum_{n=0}^{\infty} \left( A_n a_0^n + \frac{B_n}{a_0^{n+1}} \right) P_n = f(\phi) = \sum_{n=0}^{\infty} a_n P_n.$$

We balance coefficients to get

$$A_n a_0^n + \frac{B_n}{a_0^{n+1}} = a_n.$$ 

Now let $S(\phi) = \sum_{n=0}^{\infty} a_n P_n(\cos \phi)$ be the BC at $r=b$.

Then $\Phi(b, \phi) = \sum \left( A_n b^n + \frac{B_n}{b^{n+1}} \right) P_n = \sum_{n=0}^{\infty} s_n P_n.$

So $A_n b^n + \frac{B_n}{b^{n+1}} = s_n.$
(3) (Continued) For each \( n \) we have two simultaneous linear equations to solve for \( A_n \) and \( B_n \). The result is

\[
A_n = \frac{g_n b^{n+1} - f_n G^{n+1}}{b^{2n+1} - G^{2n+1}}, \quad B_n = (ab(392,754),(466,801))^{n+1} \frac{f_n b^n - g_n G^n}{b^{2n+1} - G^{2n+1}}.
\]

In the present case, \( \Phi(r, \phi) = f(r) = \Phi_1 e^{i\phi} \)
and \( \Phi(b, \phi) = g(\phi) = \Phi_2 e^{i\phi} \), so \( f = \Phi_1 \), \( g = \Phi_2 \)
and all other \( \Phi \)'s and \( G \)'s are zero. Then

\[
A_1 = -\frac{\Phi_1 a^2}{b^3 - a^3}, \quad B_1 = (ab)^2 \frac{\Phi_1 b}{b^3 - a^3},
\]

and

\[
A_2 = \frac{\Phi_2 b^3}{b^5 - a^5}, \quad B_2 = -(ab)^3 \frac{\Phi_2 G^2}{b^5 - a^5}.
\]

The solution can then be put into the form

\[
\Phi(r, \phi) = \frac{\Phi_1 a^2 b}{b^3 - a^3} P_1(\cos \phi) \left( -\frac{r}{b} + \frac{b}{r^2} \right) + \frac{\Phi_2 b^3 a}{b^5 - a^5} P_2(\cos \phi) \left( \frac{r^2}{b^2} - \left( \frac{a}{b} \right)^3 \right).
\]

From the given boundary conditions, we could have guessed initially that only \( P_1 \) and \( P_2 \) would appear in the solution.

**Challenge Problem**

(a) For axisymmetric functions in cylindrical coordinates,

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi}.
\]
where \( L = \frac{1}{\sin \phi} \frac{d}{d \phi} \left( \frac{d}{d \phi} (\sin \phi \frac{d}{d \phi} \psi) \right) \).

We have \( L \psi_n (r \sin \phi) = \frac{1}{\sin \phi} \left\{ \frac{d}{d \phi} (\sin \phi \frac{d}{d \phi} \psi_n (r \sin \phi)) \right\} \).

We know that \( \psi_n (r \sin \phi) \) is a solution of Legendre's equation with \( \lambda = n(n+1) \), so

\[
\frac{d}{d \phi} \left( \sin \phi \frac{d}{d \phi} \psi_n (r \sin \phi) \right) = -n(n+1) \psi_n (r \sin \phi),
\]

so \( L \psi_n (r \sin \phi) = -n(n+1) \psi_n (r \sin \phi) \).

(b) We try \( T(r, \phi) = \sum_{n=0}^{\infty} C_n (r) \psi_n (r \sin \phi) \). We expand the known functions \( \Gamma (r, \phi) \) and \( G (\phi) \) in Legendre polynomials:

\[
\Gamma (r, \phi) = \sum_{n=0}^{\infty} d_n (r) \psi_n (r \sin \phi), \quad \text{where} \quad d_n (r).
\]

It is known:

\[
d_n (r) = \frac{2n+1}{2} \int_0^\pi \Gamma (r, \phi) \psi_n (r \sin \phi) \sin \phi \, d\phi.
\]

\[
G (\phi) = \sum_{n=0}^{\infty} g_n \psi_n (r \sin \phi), \quad \text{where} \quad g_n \text{ is known}.
\]

\[
g_n = \frac{2n+1}{2} \int_0^{2\pi} G (\phi) \psi_n (r \sin \phi) \sin \phi \, d\phi.
\]

We substitute the expansions for \( T \) and \( \Gamma \) into the partial differential equation. We first calculate \( \frac{\partial^2}{\partial \phi^2} T \):

\[
\frac{\partial^2}{\partial \phi^2} T = \sum_{n=0}^{\infty} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} (r^2 \frac{d}{dr} C_n) \right) \psi_n (r \sin \phi) + \frac{d_n (r)}{r^2} L \psi_n.
\]

\[
= \sum_{n=0}^{\infty} \left\{ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} C_n \right) - \frac{d_n (r)}{r^2} C_n \right\} \psi_n (r \sin \phi)
\]
We substitute the expansions for $V^2 T$ and $T$ into the partial differential equation to get

$$\sum_{n=0}^{\infty} \left\{ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_n}{dr} \right) - \frac{n(n+1)}{r^2} C_n \right\} P_n(\cos \phi)$$

$$= -\sum_{n=0}^{\infty} \delta_n(r) P_n(\cos \phi).$$

We balance coefficients of $P_n$ to get

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_n}{dr} \right) - \frac{n(n+1)}{r^2} C_n = -\delta_n(r).$$

For each $n$, this is an ordinary differential equation to solve for $C_n(r)$. We require that $t_n$ be well-behaved at $r=0$. The second condition on $C_n$ comes from the boundary condition on the surface of the sphere:

$$T(\theta, \phi) = \sum_{n=0}^{\infty} C_n(\theta) P_n(\cos \phi) = g(\phi) \sum_{n=0}^{\infty} \delta_n P_n(\cos \phi)$$

We balance coefficients of $P_n(\cos \phi)$ to get

$$C_n(\theta) = \delta_n.$$

(c)

In general, the problem for $C_n$ may be solved with a Green's function. That is more machinery than we need for the present case. We solve the problem by finding the solution to the homogeneous problem, find a particular solution, superposing the two, and imposing the boundary conditions.
The homogeneous equation is
\[
\frac{d}{dr} \left( r^2 \frac{dC}{dr} \right) - \eta \left( r^2 \right) C = 0.
\]
This is an equation of homogeneity and the solutions easily are found to be \( r^n \), \( r^{-n} \) - something we already knew from solving the Laplace equation. So
\[
C_m = A_m r^n + B_m r^{-n}, \quad (C_m = \text{homogeneous solution})
\]
We have \( \Gamma(r, \phi) = r \Rightarrow r^2 P_2 \). \( \Rightarrow 2 \phi_2 + 8 \phi_r^2 P_2 \).
So \( \phi_0 = 0 \), \( \phi_2 = 8 \phi_r^2 \), and all other \( \phi_n \)'s = 0.
We denote the particular solutions by subscript \( \sigma \). Then
\[
C_{\sigma} = 0 \text{ for } n \neq 0 \text{ and } n \neq 2.
\]
Now we consider \( C_0 \) and \( C_2 \). We have
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_0}{dr} \right) = -\alpha
\]
\[
\frac{d}{dr} \left( r^2 \frac{dC_0}{dr} \right) = -\alpha r^2 \Rightarrow r^2 \frac{dC_0}{dr} = -\frac{\alpha r^3}{2}
\]
So \( \frac{dC_0}{dr} = -\frac{\alpha r}{2} \), \( C_0 = -\frac{\alpha r^2}{6} \).
So \( C_0 = C_0 + \frac{b_0}{r} - \frac{\alpha r^2}{6} \).

The solution must be well-behaved as \( r \to 0 \), so we require \( b_0 = 0 \). Then
\[
C_0(r) = C_0 - \frac{\alpha r^2}{6}.
\]
For \( n=2 \), we have \( \frac{d}{dr} \left( r^2 \frac{dC_2}{dr} \right) - 6C_2 = -\beta r^4 \).
We try \( C_0 = k r^4 \). The equation yields
\[ C_0 = \frac{\Theta r^4}{14} \]

Then
\[ C_2(r) = C_2 r^2 + \frac{b_2}{r^3} - \frac{\Theta}{14} r^4. \]

For the solution to be well-behaved at \( r = 0 \), we must have \( b_2 = 0 \). Then
\[ C_2(r) = C_2 r^2 - \frac{\Theta}{14} r^4. \]

For all other \( n \), \( C_n = 0 \), and
\[ C_n(r) = C_n r^n + \frac{b_n}{r^n}. \]

For the solution to be well-behaved at \( r = 0 \), we must take \( b_n = 0 \).

Summary: \[ C_n(r) = \begin{cases} C_0 - \frac{\Theta r^2}{6}, & n = 0 \\ C_2 r^2 - \frac{\Theta}{14} r^4, & n = 2 \\ C_n r^n, & n \neq 0, n \neq 2. \end{cases} \]

The final step is to impose the initial conditions, which are \( C_0 (\phi) = 0 \). In our case,
\[ C_0 (\phi) = \frac{1}{2} \sin^2 \phi = \frac{1}{2} \sin \left( \phi - \frac{\pi}{2} \right) \] as shown in Problem (1).

So \( C_0 = \frac{1}{2} \), \( C_2 = -\frac{1}{2} \), all other \( C_n = 0 \). Now we see that only \( C_0(r) \) and \( C_2(r) \) are non-zero.
\[ n=0: \quad c_0(r) = G_0 - \frac{ar^2}{6}, \quad c_0(a) = G_0 = \frac{2G}{3\sigma}, \quad G_0 = \frac{\sigma a^2}{6} + \frac{2}{3}\sigma \\
\text{and} \quad c_0(r) = \frac{a}{6} (G^2r^2) + \frac{2}{3}\sigma \]

\[ n=2: \quad c_2(r) = G_2 r^2 - \frac{b}{14} r^4, \quad c_2(a) = G_2 = -\frac{2}{3}\sigma \]

So \[ G_2 G^2 - \frac{b}{14} G^4 = -\frac{2}{3}\sigma, \]

And \[ c_2(r) = \frac{a}{14} r^2 (G^2 - r^2) - \frac{2}{3}\sigma \frac{r^2}{G^2}. \]

The solution is

\[ T(n) = \frac{a}{6} (G^2 - r^2) + \frac{2}{3}\sigma + \frac{c}{14} (G^2 - r^2) = \frac{2}{3}\sigma \frac{r^2}{G^2} P_2 (10\sigma). \]
Problem 1

The solution obtained by hand is defined below for Mathematica.

```mathematica
In[1]:= \[Phi][r_, \eta_] := (2/3) \Phi[0 \{1 - LegendreP[2, \eta] (r / a)^2\}]

We specify the parameter values.

In[2]:= a = 1.5 \text{ (m)}; \Phi[0 = 20 \text{ (volts)}];

Now we create a rectangular coordinate version of the potential. We work in the plane \(y = 0\).

In[3]:= \[Phi]rec[x_, z_] := \[Phi][r, \eta] /. \{\eta \rightarrow z / Sqrt[x^2 + z^2], r \rightarrow Sqrt[x^2 + z^2]\}

We make the contour plot. We use an option called RegionFunction to plot the contours only in the relevant circular domain, representing the cut of the sphere by the plane \(y = 0\). We also use the option ContourLabels -> All to put labels on the contours. As specified in the problem statement, we plot the contours for 5, 10, and 15 volts.

In[4]:= contour = ContourPlot[\[Phi]rec[x, z], \{x, -1.2 \cdot a, 1.2 \cdot a\},
{z, -1.2 \cdot a, 1.2 \cdot a\}, RegionFunction -> Function[\{x, z\}, 0 < x^2 + z^2 < a^2],
Contours -> \{5, 10, 15\}, ContourLabels -> All, ImageSize -> 300]
```

Out[4]=

On the boundary, the potential is \(\Phi (\sin \phi)^2\) for which the minimum of zero volts occurs at the poles, and the maximum of 20 volts occurs at the equator. Because the minimum and maximum values of the solution of Laplace’s equation occur on the boundary, the voltage in the interior will be between 0 and 20 volts. The contour plot above exhibits this behavior.