Assignments handed in by 6 PM on Wednesday Sept. 28 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday Sept. 29.

LECTURE SCHEDULE AND READING

Section in Class Notes | Date | Section in Text
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2. Fourier Series
2.4 Fourier Sine and Cosine Series | W Sept. 21 | 3.3
2.5 Separation of Variables Revisited | Th Sept. 22 | Chapter 2
3. Separation of Variables, Part I
3.1 Diffusion Equation | F, M Sept. 23, 26 | 8.2

PROBLEMS

2.4 FOURIER SINE AND COSINE SERIES

(1) (21 points) Consider the function \( f(x) = x + |x| \). In each of the three parts below, find explicitly the series asked for, sketch the periodically extended function represented by the series on \([-3,3]\), and show that the rate of convergence of the series is consistent with the smoothness of the periodically extended function. Tell what the series converges to at each point of the given interval.

(a) (7 points) The full Fourier series for \( f(x) \) on \(-1 \leq x \leq 1\).
(b) (7 points) The Fourier sine series for \( f(x) \) on \(0 \leq x \leq 1\).
(c) (7 points) The Fourier cosine series for \( f(x) \) on \(0 \leq x \leq 1\).

2.5 SEPARATION OF VARIABLES REVISTED

(2) (14 points) A squash player is attempting to warm up a squash ball before play by immersing it in a basin of hot water. The initial temperature of the ball is 20 °C and the water in the basin is at 50 °C. For purposes of this problem, take the squash ball to be solid rubber, with diameter 5 cm, and with a thermal diffusivity of \(5 \times 10^{-8} \text{ m}^2/\text{s}\). The player soaks the ball for one minute. Use what you know about diffusion times to comment on the efficacy of this process.

3.1 DIFFUSION EQUATION

(3) (65 points) Consider the initial value problem given below for \( T(x,t) \).

\[
\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} + \gamma, \quad 0 < x < L, \ t > 0, \text{with} \ T(0,t) = 0, \ T(L,t) = \alpha L, \text{and} \ T(x,0) = \alpha x.
\]

(CONTINUED NEXT PAGE)
Here \( \gamma, \alpha, \) and \( D \) are all positive constants. This is the equation for transient heat conduction in a bar of length \( L \), with the ends maintained at prescribed temperatures, with a heat source of strength \( \Gamma = \rho C \gamma \) (where \( \rho \) is the density and \( C \) is the specific heat), and with an initial linear distribution of temperature.

(a) (5 points) Follow the procedure given in class, and split the solution \( T \) into a steady-state part \( T_s \) and a transient part \( \hat{T} \). Give a complete formulation (equation, boundary conditions, and, for the transient, an initial condition) for the determination of \( T_s \) and \( \hat{T} \).

(b) (10 points) Find the steady-state solution \( T_s \).

(c) (10 points) Find the transient solution \( \hat{T} \). You may make use of any results obtained in class, but be sure to explain your work.

(d) (5 points) Verify that your solution \( T \) approaches \( T_s \) as time \( t \) goes to infinity.

(e) (10 points) Sometimes we want to have more information than the simple result of (b) gives us – for example, an approximate solution which shows the approach to steady state. Show that for large times, the series solution for the transient \( \hat{T} \) reduces to a simple one-term approximation. Estimate how large the time must be for this approximation to be valid. (Your estimate will be in terms of the parameters of the problem.)

(f) (10 points) For the following parameter values, find the time at which the temperature and the steady-state temperature differ by less than \( \pm 2 \) °C throughout the bar: \( L = 0.1 \) m, \( \alpha = 2000 \) °C/m, \( D = 0.2 \times 10^{-3} \) m\(^2\)/s, and \( \gamma = 10 \) °C/s.

(g) (10 points) For the parameter values given in (f), use Mathematica to construct some graphs of \( T \) versus \( x \) for various values of time \( t \). The two principal practical problems you must solve to make this useful are (1) the determination of the number of terms in the series needed for reasonable accuracy, and (2) the determination of a set of time values for the graphs so that you get a good overview of the entire diffusion process.

(h) (5 points) Construct a graph of the steady-state solution. Find the location and value of the maximum temperature in steady state.

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CHALLENGE PROBLEM

In this problem, you will look at a simple model for oxygen transport and consumption in a muscle cell. The muscle cell is modeled as a slab of thickness \( L = 36 \, \mu \text{m} \). The oxygen concentration \( N(x,t) \) varies with position \( x \) and time \( t \). Here \( x \) is a coordinate running across the thickness of the slab. We assume that the concentration does not depend on the \( y \) and \( z \) coordinates in the plane of the slab. The oxygen concentration on the outer edges of the cell (\( x = 0 \) and \( x = L \)) is maintained by the capillaries at a concentration \( N_0 = 6 \times 10^{-9} \, \text{mol/cm}^3 \). Within the cell, oxygen is consumed at a rate \( \Gamma \, \text{mol/(cm}^3\cdot\text{s)} \). The transport of oxygen within the cell is by diffusion. The diffusive flux is \( -D \frac{\partial N}{\partial x} \), where you may take the diffusivity \( D \) to be constant with the value \( D = 2 \times 10^{-5} \, \text{cm}^2/\text{s} \).

(a) (30 points) In this part of the problem (and only this part), you will consider a general, transient three-dimensional transport problem. In that case the oxygen flux is \( F = -D \nabla N \). By carrying out an oxygen balance for an arbitrary sub-volume of the region in which the transport takes place, show that the concentration \( N \) satisfies the equation

\[
\frac{\partial N}{\partial t} = D \nabla^2 N - \Gamma.
\]

Show that for the one-dimensional problem described above, this reduces to

\[
\frac{\partial N}{\partial t} = D \frac{\partial^2 N}{\partial x^2} - \Gamma.
\]

(b) (10 points) Assume that the initial oxygen concentration in the cell is \( N(x,0) = N_0 \), the same as the concentration maintained on the cell boundaries. Give a complete statement of the formulation of the boundary value problem for \( N(x,t) \), including the boundary conditions, the initial condition, and the equation you derived in part (a).

(c) (30 points) Find the steady-state solution of the problem you formulated in part (b). For resting skeletal muscle, a typical consumption rate is \( \Gamma = 1 \times 10^{-9} \, \text{mol/(cm}^3/\text{s)} \). Use your steady-state solution to estimate the minimum oxygen concentration in the cell for this consumption rate. Give some discussion of your answer.

One of the defects of our over-simplified model is that it can predict negative values of oxygen concentration when the consumption rate is high. Again use your steady-state solution and show that this will happen when the consumption rate is \( \Gamma = 1 \times 10^{-7} \, \text{mol/(cm}^3/\text{s)} \), which is a typical value for a state of heavy exercise.

(d) (30 points) In this part of the problem you are asked to solve the full transient problem formulated in part (b). Do this for initial concentration equal to boundary concentration (\( N_0 = 6 \times 10^{-9} \, \text{mol/cm}^3 \)), and for the \( \Gamma \) value given above for a state of heavy exercise. Use your solution to find the time at which the model predicts the oxygen concentration will first go negative in the cell. Can you think of a way of modifying the oxygen consumption term so that the model no longer predicts negative concentrations?