Assignments handed in by 6 PM on Wednesday September 21 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday September 22 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

**LECTURE SCHEDULE AND READING**

<table>
<thead>
<tr>
<th>Section in Class Notes</th>
<th>Date</th>
<th>Section in Text</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2 Convergence of Fourier Series</td>
<td>W, Th, F Sept. 14,15, 16</td>
<td>3.2, 3.4, 3.5</td>
</tr>
<tr>
<td>2.3 Orthogonality</td>
<td>M Sept. 19</td>
<td>2.3.6</td>
</tr>
</tbody>
</table>

**PROBLEMS**

### 2.2 CONVERGENCE OF FOURIER SERIES

In class, we showed that the Fourier series for $f(x) = x^2$ on $-L \leq x \leq L$ is given by

$$
\frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{n^2 \pi^2} (-1)^n \cos \left( \frac{n\pi x}{L} \right).
$$

This formula will be useful in some of the problems below.

1. **(20 points)**
   
   **(a) (10 points)** Find the Fourier series for $f(x) = x$ on $-1 \leq x \leq 1$ by calculating the coefficients directly from the defining formulas.

   **(b) (10 points)** Find the Fourier series for $f(x) = x$ on $-1 \leq x \leq 1$ by differentiating termwise the series for $f(x) = x^2$, and show that you get the same result.

2. **(10 points)** Use the series for $f(x) = x^2$ and the series for $f(x) = x$ found in problem (1) to find the Fourier series for $f(x) = 2x^2 - 3x + 1$ on $-1 < x < 1$.

3. **(15 points)** By integrating termwise the series for $f(x) = x^2$ and by using your series from problem (1), find the Fourier series for $f(x) = x^3$ on $-1 < x < 1$.

4. **(15 points)** Consider the Fourier series of $p(x) = a + bx + cx^2 + dx^3 + ex^4$ on $-1 \leq x \leq 1$. What is the most general function of this form for which the Fourier series converges at least as rapidly as $1/n^4$?

### 2.3 ORTHOGONALITY

5. **(15 points)**
   
   **(a) (5 points)** Show that the following two vectors (where $i, j, k$ are the usual rectangular Cartesian unit vectors) are orthogonal: $E_1 = i + 3j, E_2 = 3i - j + k$.

   Find a non-zero vector $E_3$ which is orthogonal to both $E_1$ and $E_2$.

(Continued on next page)
(b) (5 points) For the given vector \( \mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k} \), use orthogonality to find coefficients \( C_1, C_2, \) and \( C_3 \) such that \( \mathbf{A} = C_1 \mathbf{E}_1 + C_2 \mathbf{E}_2 + C_3 \mathbf{E}_3 \). Verify that 
\[ |\mathbf{A}|^2 = C_1^2 |\mathbf{E}_1|^2 + C_2^2 |\mathbf{E}_2|^2 + C_3^2 |\mathbf{E}_3|^2. \]

(c) (5 points) By dividing each \( \mathbf{E} \) by its length, form a set of three normalized vectors \( \mathbf{e}_1, \mathbf{e}_2, \) and \( \mathbf{e}_3 \). Use orthogonality to find coefficients \( c_1, c_2, \) and \( c_3 \) such that \( \mathbf{A} = c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3 \). Verify that 
\[ |\mathbf{A}|^2 = c_1^2 + c_2^2 + c_3^2. \]

(6) (25 points) In class we discussed Parseval’s theorem, which says that if a real-valued \( f(x) \), on \([-L, L]\), has the Fourier series
\[ a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right), \]
then
\[ \int_{-L}^{L} f(x)^2 \, dx = 2L a_0^2 + L \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \]

We didn’t establish the conditions for which this is true, but it is known that a sufficient condition is that \( f(x) \) is piecewise continuous on \([-L, L]\) (the theorem is actually true for a much more general class of functions).

(a) (10 points) Find the Fourier series on \( -1 \leq x \leq 1 \) of \( f(x) = 3 + 2x^2 - x^4 \). Use this series and Parseval’s theorem to prove that
\[ \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}. \]

(b) (10 points) Use Mathematica and the series of part (a) to calculate a numerical approximation to \( \pi \). Find the minimum number of terms required to reproduce the digits 3.14159.

(c) (5 points) Use the Sum function in Mathematica to see if Mathematica knows the exact result for this series.

CHALLENGE PROBLEM

In this problem you will explore the concept of completeness and the connection of this concept with Parseval’s Theorem. The context here will be more general than in class, in that we will consider arbitrary sets of real orthogonal functions. A useful reference for this problem is Chapter 6 of *Fourier Analysis and Boundary Value Problems*, E. A. González-Velasco, Academic Press, 1995, San Diego (one of the course references on reserve in Carlson).

Suppose we have a set of functions \( \varphi_k(x), k = 1, 2, \ldots \) defined on \( a \leq x \leq b \). In what follows, you may assume these functions to be well-behaved (e.g., \( C^5 \)). Suppose that the functions have the following properties: they are orthogonal, 
\[ \int_{a}^{b} \varphi_m(x) \varphi_n(x) \, dx = 0 \text{ for } m \neq n, \]
and they are normalized, \( \int_a^b (\varphi_m(x))^2 \, dx = 1 \). We call such a set orthonormal. Given a real integrable function \( f(x) \) on \([a,b]\), we may calculate the Fourier coefficients with respect to the \( \varphi \)'s:

\[
f_m = \frac{\int_a^b f(x)\varphi_m(x) \, dx}{\int_a^b (\varphi_m(x))^2 \, dx} = \frac{1}{b-a} \int_a^b f(x)\varphi_m(x) \, dx.
\]

In terms of our inner product notation, \( f_n = (f, \varphi_n) \). We call \( f_n \) the component of \( f \) with respect to \( \varphi_n \). We also define the norm of \( f \) as

\[
\|f\| = \sqrt{(f, f)}.
\]

(a) (20 points) Using what you know about the basis functions in a Fourier series, show that an example of such a set on the interval \([-1,1]\) is given by

\[
\varphi_1 = 1/\sqrt{2}, \quad \varphi_2 = \sin(\pi x), \quad \varphi_3 = \cos(\pi x), \quad \varphi_4 = \sin(2\pi x), \quad \varphi_5 = \cos(2\pi x), \quad \varphi_6 = \sin(3\pi x), \quad \ldots
\]

\[
\varphi_{2k} = \sin(k\pi x), \quad \varphi_{2k+1} = \cos(k\pi x), \quad \ldots
\]

A fundamental question about an orthonormal set is whether it is complete – that is, whether it contains enough functions to form a basis for a given class of functions – for example, piecewise smooth. As a very simple example from vector algebra, consider the set of unit vectors in rectangular coordinates: \( \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \). These vectors are pairwise orthogonal and all have unit length, so the set is orthonormal. Furthermore we can represent any vector in three space as a linear combination of these vectors. It is reasonable to call such a set complete. Now consider the set \( \{\mathbf{i}, \mathbf{j}\} \). This set is also orthonormal, but we do not call it complete in three space because there are an infinity of vectors which cannot be represented as a linear combination of these vectors (for example the vector \( \mathbf{V} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \), or any other vector containing a nonzero \( z \)-component). From our knowledge of Fourier series, we know that in the set of \( \varphi \)'s given above, we need all of the functions in order to represent an arbitrary function. If, for example, we omit \( \varphi_1 \) from the set, then we can no longer represent any function which has a non-zero mean on the interval \([-1, 1]\). We now explore further this question of completeness.

(b) (40 points) Let \( \{\varphi_i\} \) be an orthonormal set on the interval \([a, b]\). Show that for any integrable \( f \)

\[
(f, f) \geq \sum_i f_i^2,
\]

where \( f_i = (f, \varphi_i) \) is the ith Fourier coefficient of \( f \) wrt to the \( \varphi \)'s. This is called Bessel’s inequality. Show from this result that the series on the right of the inequality converges. (Note that if we had equality, we would have Parseval’s theorem: \( (f, f) = \sum_i f_i^2 \).)
(c) (40 points) Now we define completeness. We will see that completeness and Parseval’s Theorem are closely connected. We say that the set \( \{\varphi_i\} \), orthonormal on \([a, b]\), is complete if, for any integrable \( f \), \( (f, \varphi_i) = 0 \) for all \( i \) implies that \( (f, f) = 0 \). That is the set is complete if the only function orthogonal to every \( \varphi_i \) is a function with zero norm. Prove that if Parseval’s Theorem holds for every integrable \( f \) on \([a, b]\), then \( \{\varphi_i\} \) is complete on \([a, b]\).