ME 163 SPRING 2000
FINAL EXAM REVIEW

WEB SITE

Any changes in office hours or any corrections, comments or hints on the review problems will be posted on the web site. Please check it frequently between now and the exam. The address of the web site is http://www.me.rochester.edu/courses/ME163.

TIME AND PLACE OF EXAM

The exam will be on Saturday May 6, 4:00 - 7:00 PM in Morey 321 (the regular MWF classroom).

MATERIAL COVERED BY EXAM

The exam will cover everything in the course. It will cover all of the material of homework assignments #1-11, and the project. It will cover the following sections in the text: 1.1-1.3, 1.5, 2.1-2.4, 2.6, 3.2-3.4, 4.1-4.9, 4.11-4.12, 5.1-5.2, 12.2-12.4, 12.6.

The exam will be open book and notes - any reference material may be used, but you may not exchange reference material during the exam. A calculator will be useful for some of the problems.

OFFICE HOURS BEFORE THE EXAM

My office hour schedule is given below. Any changes will be posted on the web site. You are welcome to come at other times, but you might want to call or E-mail first to make sure that I will be there (x54078; clark@me.rochester.edu).

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PRACTICE FINAL EXAM

The last pages of this handout are the final exam and solutions from 1999. After you have finished your basic review, try to work this exam in three hours as practice and use the results to identify any areas that need further study. Note that the exam last year did not include material on
conservative periodic solutions and limit cycles (material covered in the last four lectures this year). The review problems below do have problems on these topics.

**SUGGESTED REVIEW PROBLEMS**

You should review the homework problems and the examples given in class. Also try to work all of the problems given here. Any corrections, additional hints or comments on these problems will be posted on the web site. Most of these problems are from *Fundamentals of Differential Equations and Boundary Value Problems*, by R. Kent Nagle and Edward B. Saff.

**FIRST ORDER EQUATIONS**

Basic Concepts, Solution Curves and Direction Fields

1. Show that \( y^2 + x - 3 = 0 \) is an implicit solution of \( \frac{dy}{dx} = -1/(2y) \).
2. Consider the initial value problem \( y \frac{dy}{dx} - 4x = 0 \), \( y(x_0) = y_0 \). What does the basic existence and uniqueness theorem tell you about the solution of this problem? What if \( y_0 = 0 \)?
3. Sketch the direction field for the equation \( \frac{dy}{dx} = 1 - \frac{y}{4} \). From your sketch, what can you say about the solutions of this equation as \( x \to \infty \)?

Separable Equations

4. Determine whether the following equation is separable: \( \frac{dy}{dx} = \frac{ye^{x+y}}{x^2 + 2} \).
5. Solve \( \frac{dy}{dx} = \frac{1-x^2}{y^4} \), with \( y(1) = 1 \). (Answer: \( y(x) = \left(5x - \frac{5}{3}x^3 - \frac{2}{3}x^5\right)^{\frac{1}{3}} \).)
6. Solve the initial value problem \( \frac{dy}{dx} = (1 + y^2) \tan(x) \), \( y(0) = \sqrt{3} \). (Answer: \( y(x) = \tan\left[\frac{\pi}{2} - \log(\cos x)\right] \).)
7. A bank account pays interest at a rate \( r \), compounded continuously. The initial balance is zero, but principal is added at a rate \( \Gamma \), also continuously. If \( \Gamma = $10,000 \) per year and \( r \) is 10% per year, how long will it take for the account to reach $100,000 ? (Answer: \( 10 \log(2) = 6.93 \) years.)
8. Find an implicit solution of \( \frac{dV}{dt} = 1 - V^4 \), \( V(0) = 0 \). At what time will \( V = 0.9 \)? (Answer: \( t = 1.103 \).)

Exact Equations

9. Classify the following equation as separable, linear, exact, or none of those:
\[ y^2 dx + (2xy + \cos y)dy = 0 \].
(10) Verify that the following equation is exact, and then find the solution satisfying
\[ y(0) = 0: \quad e^t (y - t)dt + (1 + e^t)dy = 0. \] (Answer: \[ y(t) = \frac{1 + (t - 1)e^t}{1 + e^t}. \])

(11) Find an implicit solution of the following initial value problem:
\[ (1/x + 2y^2x)dx + (2yx^2 - \cos y)dy = 0, \quad y(1) = \pi. \] (Answer: \[ \log(x) + x^2 y^2 - \sin(y) = \pi^2. \])

Linear Equations

(12) Determine whether the following equation is linear, separable, neither or both:
\[ x \frac{dx}{dt} + t^2 x = \sin(t). \]

(13) Find the general solution of
\[ \frac{dy}{dx} - y = 3x. \] (Answer: \[ y(x) = Ce^x - 3x - 3. \])

(14) Solve the following initial value problem:
\[ \sin(x) \frac{dy}{dx} + y \cos(x) = x \sin(x), \quad y\left(\frac{\pi}{2}\right) = 2. \] (Answer: \[ y(x) = 1 + \csc x - x \cot x. \])

(15) Suppose brine containing 2 kg of salt per liter flows into a tank initially filled with 500 liters of water containing 50 kg of salt. The brine enters the tank at a rate of 5 liters per minute. The mixture in the tank is kept uniform by stirring, and it flows out at the rate of 5 liters per minute. Find the concentration of salt in the tank, in kg per liter, after 10 minutes. (Answer: 0.281 kg/L.)

(16) On a hot Saturday morning while people are working inside a building, the air conditioner keeps the temperature inside at 24°C. At noon the air conditioner is turned off and the people go home. The temperature outside is a constant 35°C the rest of the afternoon. If the time constant for the building is \( k^{-1} = 4 \) hr, what will the temperature inside the building be at 2:00 PM? At 6:00 PM? When will the temperature inside the building reach 27°C? (Answer: 1:16 PM.)

Euler Method for Numerical Solution

(17) Use Euler’s method with step size \( h = 0.1 \) to approximate the solution to the initial value problem
\[ \frac{dy}{dx} = x - y^2, \quad y(1) = 0, \] at the points \( x = 1.1, 1.2, 1.3, 1.4, \) and 1.5.
(Answer: \( y(1.5) = 0.564371 \).)

LINEAR SECOND ORDER EQUATIONS

Basic Concepts

(18) Consider the initial value problem
\[ e^y y'' - \frac{y'}{x-3} + y = \log(x), \quad y(1) = \alpha, \quad y'(1) = \beta. \]
What interval of existence and uniqueness are we guaranteed by the basic theorem on existence and uniqueness?
For the following problem, verify that $y_1$ and $y_2$ are linearly independent solutions of the equation. Find the general solution of the equation, and find the solution which satisfies the given initial conditions. $ty'' - (t + 2)y' + 2y = 0$, $y_1(t) = e^t$, $y_2(t) = t^2 + 2t + 2$, $y(1) = 0$, $y'(1) = 1$.

(Answer: $y(t) = 5e^{(t-1)} - (t^2 + 2t + 2)$)

Homogeneous Constant Coefficient Equations

(20) Find the general solution of $y'' + y - y = 0$.

(Answer: $y(x) = e^{-x}(C_1 e^{\frac{\sqrt{2}x}{2}} + C_2 e^{-\frac{\sqrt{2}x}{2}})$)

(21) Find the general solution of $4y'' - 4y' + y = 0$.

(Answer: $y(x) = (C_1 + C_2 x)e^{\frac{\sqrt{2}x}{2}}$)

(22) Solve the initial value problem $y'' - 2y' - 2y = 0$, $y(0) = 0$, $y'(0) = 3$.

(Answer: $y(x) = \sqrt{3}e^x \sinh(\sqrt{3}x)$)

(23) Find the general solution of $y'' - 6y' + 10y = 0$.

(Answer: $y(x) = e^{3x}(C_1 \cos x + C_2 \sin x)$)

(24) Solve the initial value problem $y'' + 2y' + 7y = 0$, $y(0) = 1$, $y'(0) = -1$.

(Answer: $y(x) = e^{-x} \cos(4x)$)

(25) Find the general solution of $y''' - y'' + y' + 3y = 0$.

(Answer: $y(x) = C_1 e^{-x} + (C_2 \cos(\sqrt{2}x) + C_3 \sin(\sqrt{2}x)) e^x$)

Unforced Vibrations

(26) A 5-kg mass is suspended from a spring with $k = 80$ N/m. The mass is displaced 1 m below equilibrium and given a velocity of 4 m/s upward. There is no damping in the system.

(a) Use conservation of energy to find the maximum displacement. (Answer: $\sqrt{2} = 1.414$ m). What is the answer if the initial velocity is 4 m/s downward?

(b) Solve the differential equation to find $x(t)$. (Answer: $x(t) = -\cos(4t) + \sin(4t)$)

(27) Consider a damped spring mass system with $m = 5$ kg, $b = 10$ N s/m, and $k = 85$ N/m. Displacements are measured from equilibrium as usual. Find the maximum displacement from equilibrium if the initial conditions are (a) $x(0) = 3$ m, $\dot{x}(0) = 7$ m/s (Answer: 3.39 m), (b) $x(0) = 3$ m, $\dot{x}(0) = -7$ m/s. (Answer: 3 m.)

Basic Concepts for Inhomogeneous Equations

(28) A certain second order, linear inhomogeneous equation for $y(t)$ is known to be satisfied by the following three functions: (a) $3e^t + 2t^2 + 4t + 4 + e^{-t}$, (b) $t^2 + 2t + 2 + e^{-t}$, (c) $2\cosh t$. Show that the general solution is given by $C_1 e^t + C_2 (t^2 + 2t + 2) + 2\cosh(t)$. 

Inhomogeneous Constant Coefficient Equations

(29) Find a particular solution of $x'' - x = 3e^{-2t}$. (Answer: $\frac{1}{2}e^{-2t}$.)

(30) Find a particular solution of $y'' + y' + y = 2\cos(2x) - 3\sin(2x)$. (Answer: $y = \sin(2x)$.)

(31) Solve the initial value problem $y'' + y = 2e^{-x}$, $y(0) = 0$, $y'(0) = 0$. (Answer: $y(x) = e^{-x} - \cos x + \sin x$.)

(32) Find a particular solution of $y''' - y'' + y = \sin x$. (Answer: $y(x) = \frac{1}{2}\cos x + \frac{2}{5}\sin x$.)

(33) What is the correct form for a particular solution of $y'' + 2y' + 5y = e^{-x}\sin(2x)$. (Answer: $y_p = x[A\cos(2x) + B\sin(2x)]e^{-x}$.)

Sinusoidally Forced Vibrations

(34) An object of mass 1 slug is attached to a spring with spring constant $k = 10$ lb/ft. At time $t = 0$, an external force $F(t) = 3\cos(4t)$ is applied to the mass. The damping constant is $b = 1$ lb s/ft. What is the amplitude of the oscillation about equilibrium when the system has reached a steady oscillation state? (Answer: $0.416$ ft.)

(35) A precision rotating machine runs at 3600 rpm. If the machine is off-balance, it will vibrate. It is desired to make a simple vibration detector from a spring-mass system. The detector will be placed on the rotating machine, and we desire a maximum response from the detector at the frequency corresponding to the rotation of the machine. The mass in the detector is 2 kg. The damping is negligible. What should the spring constant be? (Answer: $k = 284,245$ N/m.)

Equidimensional Equation

(36) Solve the initial value problem $x^2y'' + 7xy' + 5y = 0$, $y(1) = -1$, $y'(1) = 13$. (Answer: $y(x) = (2/x) - (3/x^5)$.)

(37) Find the general solution of $(x - 2)^2y'' - 7(x - 2)y' + 7y = 0$. (Answer: $y(x) = C_1(x - 2) + C_2(x - 2)^7$.)

Reduction of Order

(38) Consider the equation $y'' + 2y' - 15y = 0$. Verify that $y_1 = e^{3x}$ is a solution. Use reduction of order to find a second solution. (Answer: $y_2 = e^{-5x}$.)

(39) Consider the equation $xy'' - (x + 1)y' + y = 0$. Verify that $y_1 = e^x$ is a solution. Use reduction of order to find a second solution. (Answer: $y_2 = x + 1$.)
Variation of Parameters

(40) Use variation of parameters to find a general solution of \( y'' + 2y' + y = e^{-x}. \)
(Answer: \( y(x) = (C_1 + C_2 x + \frac{1}{2} x^2)e^{-x}. \) )

(41) Use variation of parameters to find a general solution of \( y'' + 4y' + 4y = e^{-2x}\log(x). \)
(Answer: \( y(x) = (C_1 + C_2 x + \frac{2}{x}(2\log(x) - 3))e^{-2x}. \) )

SYSTEMS OF EQUATIONS

Basic Concepts

(42) Convert each equation below to a system of first-order equations.
(a) \( x'' + 3x' - x^2 = 5\cos(7t). \)
(b) \( x' = 0. \)
(c) \( y'' + y' = 3x - y^2. \)
(d) \( y''' + 3y'' + 3y' + y = 0. \)

Stability for Linear Autonomous Systems of Two Equations

(43) Classify the equilibrium at the origin for the following system: \( \dot{x} = 3x, \dot{y} = 3y. \)
(Answer: unstable node.)

(44) Classify the equilibrium at the origin for the following system: \( \dot{x} = 6x - y, \dot{y} = 8y. \)
(Answer: unstable node.)

(45) Classify the equilibrium at the origin for the following system:
\( \dot{x} = 2x - 2y, \dot{y} = 10x + 6y - 16. \) (Answer: unstable spiral.)

(46) Find and classify the equilibrium for the following system:
\( \dot{x} = 2x + y + 9, \dot{y} = -5x - 2y - 22. \) (Answer: center.)

(47) Find and classify the equilibrium for the following system:
\( \dot{x} = 2x + 4y + 4, \dot{y} = 3x + 5y + 4. \) (Answer: saddle.)

Stability for Nonlinear Autonomous Systems of Two Equations

(48) Analyze the stability of the equilibrium point at the origin for the following system:
\( \dot{x} = x + 5y - y^2, \dot{y} = -x - y - y^2. \) (Answer: the linearized equations have a center, so there is no conclusion from the theorem for the nonlinear equations.)

(49) Analyze the stability of the equilibrium point at the origin for the following system:
\( \dot{x} = e^{x+y} - \cos x, \dot{y} = \cos y + x - 1. \) (Answer: saddle, hence unstable.)

(50) Find all of the equilibrium points and analyze the stability of each for the following system:
\( \dot{x} = 16 - xy, \dot{y} = x - y^3. \) (Answer: stable node, saddle.)
Conservative Periodic Systems

**51)** Consider the equations \( \ddot{x} + x + x^5 = 0 \).
(a) Convert this equation to a system of two first order equations.
(b) Find an energy function for the system, and show directly that it is constant. (Answer: \( E = \frac{1}{2}v^2 + \frac{1}{2}x^2 + \frac{1}{5}x^7 \), where \( v = \dot{x} \).)
(c) Find the equation of the orbit passing through the initial point \( x = -2, v = 3 \).
(d) Estimate the period of the motion for the initial conditions \( x(0) = 0.1, v(0) = 0 \).

Limit Cycles

**52)** Consider the system of equations \( \dot{x} = x - y - 4x(x^2 + y^2), \quad \dot{y} = x + y - 4y(x^2 + y^2) \).
(a) Introduce the polar radial coordinate \( r \) defined by \( r^2 = x^2 + y^2 \). Obtain a differential equation for \( r \). (Answer: \( r = r(1 - 4r^2) \).)
(b) From your equation of part (a), infer that this system has a stable limit cycle \( r = 1/2 \), and that every initial point in the plane is attracted to this limit cycle.
(c) By deriving an equation for \( \dot{\theta} \), where \( \theta \) is the polar angular coordinate, show that the period of the limit cycle is \( 2\pi \).