(1) (a) For $t=0$, $b=b$, so $(b_0-b) = b_0$ which is greater than the death rate $b$. Hence the population increases.

(b) As $t \to \infty$, $b(t) \to b$, which just secures the death rate.

\[ \frac{dp}{dt} = \text{birth rate - death rate} \]

\[ = b(t)P - bP = bP + (b_0 - b)e^{at}P - bP \]

\[ = (b_0 - b)e^{at}P \]

It is separable:

\[ \frac{dp}{P} = (b_0 - b)e^{at}dt \]

\[ \ln P = \frac{-(b_0 - b)}{a} + C . \]

We impose the initial condition:

\[ \ln P_0 = C - \frac{b_0 - b}{a} \]

So $C = \ln P_0 + \frac{b_0 - b}{a}$

and hence

\[ \ln P = \frac{b_0 - b}{a} + \ln P_0 \]

\[ \Rightarrow P(t) = P_0 e^{\left(\frac{b_0 - b}{a} \left(1 - e^{-at}\right)\right)} \]

As $t \to \infty$, $P(t)$ approaches the constant value

\[ P_\infty = P_0 \exp\left[\frac{b_0 - b}{a}\right] . \]

(2) $M = 2e^{-2x} - ye^{-x}$, $N = e^{-x}$. We have $\frac{\partial M}{\partial y} = -e^{-x}$

and $\frac{\partial N}{\partial x} = -e^{-x}$ so it is exact. We seek an $F$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$. The first of these is

\[ \frac{\partial F}{\partial x} = 2e^{-2x} - ye^{-x} \]

\[ \Rightarrow F = -e^{-2x} + ye^{-x} + g(y) . \]

We substitute this into $N = \frac{\partial F}{\partial y}$

\[ e^{-x} = \frac{\partial}{\partial y}\left(-e^{-2x} + ye^{-x} + g(y)\right) \]

\[ = e^{-x} g'(y) . \]

\[ \therefore g'(y) = 0 \]

\[ \Rightarrow F = -e^{-2x} + ye^{-x} \]
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(2) (cont'd) The solution is \(-e^{-2x} + ye^{-x} = C\). The initial condition gives \(y = 1\), so \(-e^{-2x} + ye^{-x} = \frac{1}{2}\) and \(y(x) = e^{-x} - e^{-2x}\).

(3) Standard form: \(\frac{dy}{dx} + \frac{1}{x} y = \frac{\cos(x)}{x}\).

The integrating factor is \(u = e^{\int \frac{1}{x} dx} = e^\ln x = x\). Then

\[
x \frac{dy}{dx} + y = \cos(x)
\]

\[
\frac{d}{dx}(xy) = \cos(x)
\]

\[
xy = \sin(x) + C
\]

\[
y = \frac{\sin(x)}{x} + \frac{C}{x}
\]

\(y(\pi) = 1 = \frac{C}{\pi}\) so \(C = \pi\) and

\[
y(x) = \frac{\sin(x)}{x} + \frac{\pi}{x}
\]

(4) The governing equation is \(\frac{dT}{dt} = -K(T - M)\). This was solved in class for constant \(M\) to get

\[T(t) = M + (T_0 - M)e^{-Kt}\]

We take \(t = 0\) to be midnight. We have \(M = 29\) and \(T_0 = 68\).

Thus \(T(t) = 29 + (39)e^{-Kt}\). Thus we solve for the time at which this becomes equal to 50:

\[50 = 29 + 39e^{-Kt}\]

\[so \ e^{-Kt} = \frac{29}{11} \Rightarrow Kt = \ln\left(\frac{39}{29}\right) = 0.690\]

\[\Rightarrow t = \frac{0.690}{K} = 4.9523 \text{ or } 5\text{.95} \text{ min.}\]

The cars will go off at 5:57 am so there is no need for the mechanical alarm clock.
(2) First we find the exact solution. By integration we get \( y = \frac{1}{2} x^2 + C \), \( y(0) = 1 \Rightarrow C = 1 \), so \( y = 1 + \frac{1}{2} x^2 \).

Now we carry out the Euler steps: \( y_{n+1} = y_n + h x_n \)

- \( x_0 = 0, y_0 = 1 \)
- \( x_1 = 0.1, \quad y_1 = y_0 + h x_0 = 1 \)
- \( x_2 = 0.2, \quad y_2 = y_1 + h x_1 = 1 + (0.1)(0.1) = 1.01 \)
- \( x_3 = 0.3, \quad y_3 = y_2 + h x_2 = 1.01 + (0.1)(0.2) = 1.03 \)

Exact: \( y(0.3) = 1 + \frac{1}{2}(0.3)^2 = 1.045 \)