Servo control of an optical trap

Kurt D. Wulff,1,* Daniel G. Cole,2 and Robert L. Clark1

1Center for Biologically Inspired Materials and Material Systems, Duke University, Durham, North Carolina 27708, USA
2Department of Mechanical Engineering and Materials Science, University of Pittsburgh, Pittsburgh, Pennsylvania 15261, USA
*Corresponding author: kurt.wulff@duke.edu

Received 7 February 2007; revised 27 April 2007; accepted 1 May 2007; posted 2 May 2007 (Doc. ID 79871); published 3 July 2007

A versatile optical trap has been constructed to control the position of trapped objects and ultimately to apply specified forces using feedback control. While the design, development, and use of optical traps has been extensive and feedback control has played a critical role in pushing the state of the art, few comprehensive examinations of feedback control of optical traps have been undertaken. Furthermore, as the requirements are pushed to ever smaller distances and forces, the performance of optical traps reaches limits. It is well understood that feedback control can result in both positive and negative effects in controlled systems. We give an analysis of the trapping limits as well as introducing an optical trap with a feedback control scheme that dramatically improves an optical trap’s sensitivity at low frequencies. © 2007 Optical Society of America

OCIS codes: 120.0120, 140.7010, 110.0180, 120.4640.

1. Introduction

Optical traps use the momentum of light to exert forces on microscopic objects. As light is diffracted and reflected through dielectric objects, the change in momentum of the light results in a force exerted on the refracting object. This force is proportional to the power of the beam and has two components: a scattering force that acts in the direction of propagation of the light and a gradient force that acts in the direction of increasing intensity, acting to pull trapped objects toward the axis of the beam [1]. The magnitude of these forces is small, on the order of piconewtons to several hundreds of piconewtons. For example, a moderate 200 mW beam can produce a force on the order of 100 pN.

Initial experiments used optical traps as probes to observe their molecular interactions. Ashkin and Dziedzic demonstrated the nonlethal trapping of viruses and E. coli bacteria [2] as well as the manipulation of filaments inside living cells [3]. Chu’s group later observed the refolding of DNA after first using an optical trap to uncoil it [4]. As optical traps matured, force measurements became possible. Kuo and Sheetz [5] estimated the force produced by a single kinesin molecule by attaching a microsphere to the tail of a kinesin molecule, trapping it, and monitoring displacements through video analysis. Using a sensitive position detection system based on Nomarski differential interference contrast (DIC) microscopy, Svoboda et al. directly measured the step size of kinesin walking along a microtubule [6].

Construction and use of optical traps has become relatively mature. Using feedback control, a versatile optical trap can be constructed to control either the position of trapped objects (position clamp) or, alternatively, apply specified forces (force clamp). The advantages of feedback control are well established and include servo control and tracking, rejection of disturbances, and operation in the presence of uncertainty. In the case of optical traps each of these has benefits for controlling particle position or applying force, minimizing the effects of fluctuating forces that result in Brownian motion, and effective operation in the presence of (somewhat) uncertain trap characteristics. An early use of feedback control of optical traps for force measurements was done by Finer et al. [7], where a feedback control scheme was implemented using a quadrant photodiode position sensor creating a so-called position clamp [8] (position servo control) for the study of actin and myosin interactions. Wang
et al. also used a servo controlled optical trap to examine the compliance of DNA [9], modulating the power to maintain a constant position. This was a unique setup compared to directly controlling the position of the trap within the reference of the focal plane, either by moving the trap or the reference plane. Visscher et al. developed a so-called force clamp [10–12] (force servo control), which, in order to study the stall force of kinesin, maintained constant force by appropriately controlling the position of objects within the trap.

Optical trapping system inputs–outputs include a method to steer the beam, typically using a fast-steering mirror (FSM) or acousto-optical deflector (AOD), and an optical means of measuring the position of objects within the trap. Many position sensing methods have been developed. Video techniques provide direct measurement of the absolute position of trapped objects [13–15], but these measurements are slow, ultimately limiting the bandwidth of the closed-loop system. Faster techniques for measuring the position of trapped objects use interferometric techniques to measure the scattered wave, which is a function of the position of the trapped object. Two common methods are a quadrant photodiode (QPD) method, which measures the change in the angular spectrum of the scattered wave about the trapping meridian [16], and a Nomarski DIC interferometric technique, which measures the change in transmitted polarization and is directly related to the scattered wave. Both of these techniques measure the displacement of the object relative to the beam. The DIC technique is arguably the most sensitive with measurement on the order of picometers [17], but they can only sense displacements along the Wollaston shear axis, while the QPD can detect displacements on the order of nanometers in two dimensions.

Measured displacements can be fed back to an actuator to control the position of the trapped object, thus servo controlling position. The position of the trap in the specimen plane is controlled by specifying the angular spectrum at a back focal plane (BFP) of the objective. The BFP is a Fourier plane with respect to the specimen plane, so a change in the angle of incidence of the trapping beam results in a translation of the trapping position. This can be effected using an AOD, an FSM, or a spatial light modulator (SLM). AODs provide the quickest response with a bandwidth ~10–100 kHz. The primary disadvantage of AODs is that they are costly and two AODs are needed to control the trap position in two dimensions. FSMs provide a response typically on the order of kilohertz. As we will see, for most trapping applications in water this provides more than adequate bandwidth for feedback control. A diffractive optical element, such as an SLM can also control the trapping position. While the incorporation of SLMs has added a new dimension to the capabilities of optical traps [18–23] giving the ability to trap multiple objects in three dimensions, the liquid crystals used in SLMs have a long response time, ~10 ms, which are suitable for video rate applications but too slow to control an object’s position over a significant bandwidth.

While the design, development, and use of optical traps has been extensive and feedback control has played a critical role in pushing the state of the art, it is surprising to note that few comprehensive examinations of feedback control of optical traps have been undertaken [24]. It is well understood that feedback control can result in both positive and negative effects in controlled systems, however, the design methods and trade-offs made in the optical trapping systems reported in the literature are missing. Furthermore, the performance limits are still unclear. This paper will present the dynamics of a particle confined in an optical trap, the control objectives and performance limits, and the experimental results of a digitally controlled optical trap capable of being either a position or force clamp.

2. Dynamics of Trapped Objects

This section discusses the dynamics of objects confined in an optical trap. These dynamics will then be used for controller design and an analysis of the limits of performance. Objects in optical traps are subjected in a potential well that have an equivalent Hookean spring constant $k$. The trapped object is also subjected to viscous drag as well as a fluctuating force. The viscous drag is dependent upon the viscosity of the fluid, $\mu$, and the size of the object. For a sphere with a radius of $r$, the drag coefficient is well defined as $\gamma = 6\pi\mu r$. The fluctuating force, $f_\text{fluct}$, is due to high frequency collisions with molecules in the fluid resulting in Brownian motion. This force is zero-mean white noise disturbance with variance

$$\sigma_f^2 = 2\gamma k_B T,$$

where $k_B$ is the Boltzmann constant and $T$ is the absolute temperature of the fluid. It is convenient to normalize the fluctuating force by the trap stiffness, $k$, making an equivalent displacement that acts as a disturbance on the system, $\tilde{d} = f/\kappa$. The variance of the fluctuating disturbance $\tilde{d}$ is

$$\sigma_{\tilde{d}}^2 = \frac{2\gamma k_B T}{k^2}.$$  

The relationship between viscous forces opposing the motion of the object and fluctuating Brownian forces are described by the fluctuation-dissipation theorem [25].

The resultant system of the trapped object, fluid, and potential well of the optical trap is modeled as a spring-mass-damper (Fig. 1). The equation of motion is

$$m\ddot{x} + \gamma \dot{x} + kx = -m\ddot{u} - \gamma \dot{u} + k\tilde{d},$$

where $m$ is the object’s mass, $x$ is the relative position of the object with respect to the trapping laser, and $u$ is the object’s displacement from equilibrium. Figure 1 shows the range of control objectives and the importance of feedback control to achieve the precise control objectives. The primary control objectives are to control the trap position, control an object’s position over a significant bandwidth.

The performance limits are still unclear, and the design methods and trade-offs made in the optical trapping systems reported in the literature are missing. Furthermore, the performance limits are still unclear. This paper will present the dynamics of a particle confined in an optical trap, the control objectives and performance limits, and the experimental results of a digitally controlled optical trap capable of being either a position or force clamp.
is the position of the trapping laser. The absolute position of the object, \( z \), is the sum of \( x \) and \( u \). In water, the system is highly overdamped due to the dominance of viscous forces over inertial forces. In this case, the eigenvalues are two first-order poles such that

\[
\Omega = \frac{k}{\gamma} \ll \frac{\gamma}{m}.
\]

\( \Omega \) is the so-called trap bandwidth (cutoff frequency). For a 1 \( \mu \)m diameter sphere in water (\( \mu = 10^{-3} \) Pa s; \( \gamma = 10^{-8} \) Ns/m) and a moderate trap stiffness of \( k = 0.05 \) pN/nm, the bandwidth is \( \Omega = 5000 \) rad/s = 800 Hz.

The system can be converted to a continuous-time frequency-domain representation by Laplace transforming Eq. (3) (assuming zero initial conditions). A frequency-domain transfer function relates the output of the system to the input. For a discussion of linear systems see Ogata [26]. The input–output relationship for the object’s relative position is

\[
x = G_1 \dot{d} + G_2 u,
\]

where \( G_1(s) \) is the transfer function of the relative position, \( x \) to the laser position, \( u \), \( G_2(s) \) is the transfer function from the relative position to the disturbance \( \dot{d} \), and \( s \) is the Laplace variable. They are defined as

\[
G_1(s) = \frac{X(s)}{D(s)} = \frac{k}{\gamma s + k} = \frac{\Omega}{s + \Omega},
\]

\[
G_2(s) = \frac{X(s)}{U(s)} = -\frac{\gamma s}{\gamma s + k} = -\frac{s}{s + \Omega},
\]

with \( X(s) \), \( U(s) \), and \( D(s) \) representing the Laplace transforms of \( x \), \( u \), and \( \dot{d} \), respectively. The input–output relationship for the object’s absolute position is

\[
z = G_1 \dot{d} + (1 + G_2) u = G_1 (\dot{d} + u).
\]

Equation (8) illustrates that in order to reject the Brownian disturbance, \( \dot{d} \), the control input should be such that \( u = \ddot{d} \). However, direct cancellation of the Brownian disturbance \( \dot{d} \) is not possible since it is unknown and random. This fact places certain limitations on the achievable performance of the closed-loop optical trapping system.

The frequency \( \Omega \) ultimately determines the bandwidth of the closed-loop system. That is, even though a constant choice of actuator (or sensor) may limit the bandwidth of the control system, the trap bandwidth determines a practical upper limit beyond which faster actuators will have little effect. In some cases, it may be possible to push the closed-loop bandwidth beyond the trap bandwidth, but this comes at the cost of high gain controllers, which typically require a trade-off between robustness and performance.

3. Feedback Control

In many biological and physical applications the trapping laser does not directly manipulate the specimen; instead a microsphere is tethered to, for example, a motor protein and used as a handle to measure the molecular motor’s step size or stall force. Measurement of these values must be achieved in the presence of fluctuating, Brownian disturbances that drive the microsphere from the desired position. Controlling the microsphere’s position in the presence of Brownian disturbances is fundamentally a servo control problem. Improving the servo control (i.e., disturbance rejection) will lead to better force control. Since force is applied via Hooke’s law, maintaining a constant displacement, or reference position, results in a constant force.

The block-diagram configuration of the servo control system implemented in this paper is illustrated in Fig. 2. The combined actuator/plant transfer function \( (G_1G_2) \) describes the displacement from the trapping center, \( x \), due to a voltage applied to the FSM. The absolute position of the trapped object (the control variable) is \( z = x + u \). However, the trap position, \( u \) is not known due to the lack of reliable position sensors on the FSM actuator. To generate an estimate of the absolute position, \( \hat{z} \), an estimate of the trap position, \( \hat{u} \) is fed-forward; this estimate is \( \hat{z} = x + \hat{u} \). The position error, \( e \), is the difference between the estimated position, \( \hat{z} \), and the desired or reference position, \( r \). The performance objective for disturbance rejection is to minimize the error. Without loss of generality, we can consider the reference \( r = 0 \). The

![Fig. 2. Block diagram of servo controller for an optical trap.](image-url)
the ideal plant with an ideal controller (which tracked with zero steady-state error. This is the response, the error in this case, at a given frequency of response can be found using the well established tools to reduce. Controllers that minimize the root mean square (rms) response, which the action of feedback control can also produce errors, which will increase the sensitivity and trap position are correlated. Of interest in many problems is the root mean square (rms) response that the origin of the loop-gain results in a zero at the origin in the closed-loop sensitivity and perfect disturbance rejection of dc disturbances.

The sensitivity and complimentary sensitivity are

\[ S(s) = \frac{s}{s + \omega_c}, \quad T(s) = \frac{\omega_c}{s + \omega_c}. \]

Note that \( T \) is the transfer function from \( \ddot{d} \) to \( \nu \); that is, it quantifies the influence of Brownian fluctuations on the motion of the trapping beam. The transfer function from \( \ddot{d} \) to \( z \) is

\[ GS(s) = \frac{\Omega s}{(s + \Omega)(s + \omega_c)}. \]

With feedback control, the mean-squared displacement and trap position are

\[ \langle z^2 \rangle = \|GS\|^2 \|d\|^2 = \frac{1}{1 + \alpha} k_B T \],

\[ \langle u^2 \rangle = \|T\|^2 \|d\|^2 = \alpha k_B T. \]

To get a significant reduction in the mean-square beal position, the proportional gain must be large; equivalently, the crossover frequency must be greater than the trap bandwidth. However, the gain cannot, in fact, be made arbitrarily large. Additional dynamics not accounted for in this simplified model limit the available bandwidth. This fact in combination with the analytical constraint of the Bode sensitivity theorem limit the gain (see Appendix A).

The influence of measurement noise, resulting, for example, from shot noise on photodiodes or Johnson noise on the feedback resistor of the current-to-voltage amplifiers, is important as well. The transfer
function from $n$ to $z$ is $T$, which is unity within the control bandwidth. Thus, the absolute motion of the trapped object will track the noise making accurate measurements of the objects position critical. The transfer function from measurement noise $n$ to $u$ is

$$KS(s) = \frac{s + \Omega}{s + \omega_c}.$$  (19)

This transfer function is proper and therefore does not have a well defined 2-norm. As such, broadband measurement noise will be passed directly through the system. Of particular interest is the response at high frequencies, $\omega \gg \Omega$, $\omega_c$, where $|KS(j\omega)| = \alpha$. A large gain then results in motion in the trap proportional to the measurement noise. This is not necessarily a problem provided the measurement noise is small.

The relative motion of the object in the trap is important as well because if it is too large its motion could be large enough to make the trapped object exit the trap, an obviously undesirable result. The transfer function from $d$ to $x$ and $n$ to $x$ are

$$\frac{x}{d} = GS(1 + K) = \frac{(\Omega + \omega_c)s + \omega_c\Omega}{(s + \Omega)(s + \omega_c)},$$  (20)

$$\frac{x}{n} = KS(1 - G) = \frac{s}{s + \omega_c}.$$  (21)

Here we see that within the control bandwidth, the response of $x$ due to measurement noise is small within the control bandwidth, an expected result since the absolute position must track the measurement noise as discussed previously. However, above $\omega_c$, the object will track the measurement noise directly.

The response of $x$ to $d$, however, is more complex. The $d$ to $x$ transfer function has a zero at $\omega_c/(1 + \alpha)$ at a frequency below the trap or control bandwidths, $\Omega$ and $\omega_c$. At very low frequencies, less than $\omega_c/(1 + \alpha)$, the trapped object will track the Brownian fluctuations with unity gain. In the frequency range between the real poles, $-\omega_c$ and $-\Omega$, the relative motion will have its peak response, which depends upon the gain $\alpha$ and can be approximated as

$$||GS(1 + K)||_{\infty} = \begin{cases} 1 + \alpha & \alpha < 1 \\ 2/\sqrt{3} & \alpha = 1, \\ 1 + 1/\alpha & \alpha > 1, \end{cases}$$  (22)

which implies that the Brownian fluctuations are amplified in this range, but at most by $2/\sqrt{3} \approx 1.15$. A reasonable specification is to require the rms response of the relative motion to be less than the maximum allowable deflection, typically the trap radius $r$. We can approximate the mean-square response to be

$$\langle x^2 \rangle = (1 + \alpha)^2 \frac{k_B T}{k} < x_{\text{max}}^2.$$  (23)

This relationship can be rewritten to provide an upper bound on the gain $\alpha$:

$$\alpha < \frac{x_{\text{max}} - x_{\text{rms}}}{x_{\text{rms}}}.$$  (24)

where $x_{\text{rms}} = \sqrt{k_B T/k}$ is the open-loop rms response to the Brownian fluctuations. In practice, this limit is higher than might be necessary. For example, for a 1 $\mu$m diameter bead and a moderate trap stiffness of 0.05 pN/nm the upper bound is $\alpha < 55$. For traps with very low stiffness, often used to measure very small forces, the upper bound is much tighter. In this case it is insightful to consider the lower bound on the stiffness for a given gain. For the same bead and a gain of $\alpha = 1$, the lower bound on the trap stiffness is $60 \times 10^{-6}$ pN/nm, within the achievable range of trap stiffness, $10^{-7} - 1$ pN/nm [28]. Thus, the gain must be limited, particularly when very low stiffness optical traps are being used.

4. Experimental Methods

The optical trap is built with an inverted microscope (Zeiss Axiovert 200) and an Nd:YAG laser at 1064 nm (Coherent Compass 1064-500) with a maximum output of 500 mW (see Fig. 3). The trapping beam is directed into the epifluorescence port of the microscope and is introduced to the microscope’s optical path using a dichroic mirror located in the microscope’s filter cube turret. The laser beam is directed through a high numerical aperture objective (Zeiss Plan-Apochromat 63×/1.4 NA) and into the trapping plane.

The position of the trap in the specimen plane is controlled by a fast steering mirror (Newport FSM-300) positioned at a BFP of the objective. This setup translates angles introduced by the mirror into lateral displacements at the trapping plane. The FSM provides more than adequate bandwidth (as will be shown later) for feedback control and is implemented.

![Fig. 3. Experimental optical trap setup with a FSM actuator and a quadrant photodiode sensor.](Image)
in the system to ensure that the full dynamic range of the FSM is utilized.

The position sensing system, modeled after Gittes and Schmidt [16], is a quadrant photodiode (QPD), which uses the trapping laser for position sensing. This has an added advantage in that the system is intrinsically aligned. After the laser passes through the sample, the forward scattered light is collected by a high NA condenser (Zeiss Achromatic-aplanatic condenser 1.4 NA). The laser beam is then separated from the illumination by a dichroic mirror located above the condenser, which directs the laser through a collimating lens and then onto the QPD. The QPD (Advanced Phontonix, Inc.) has an active area of 17.8 mm², of which the collimating lens was selected to fill approximately 80% of this area. The resulting sensitivity can be seen in Fig. 4. The QPD is mounted on a sum/difference amplifying board (Pacific Silicon Sensor) operated in the zero bias mode. A neutral density filter prevents large laser intensities from saturating the QPD. It is important to note that this configuration detects the relative motion of the trapped object from the laser, \( x \), and not the absolute position of the particle, \( z \). The intensity and relative displacements, \( x \) and \( y \), are then anti-aliased and digitized using a dSPACE data acquisition board. A digital controller was built using MATLAB’s SIMULINK and implemented using dSPACE CONTROLDESK. The system can be operated in an open or closed-loop mode, switching between the two in real time. The main advantage of using a digital controller is the quick turnaround between designing a controller and implementing it, making the system easily configurable for specific experiments. The fast implementation time also allows multiple controllers to be tested in order to find the optimal controller.

The open-loop plant dynamics, described by \( G_1 \), were experimentally determined using the power spectrum method. A 100 mW beam was used to trap a 1 \( \mu \)m polystyrene microsphere approximately 5 \( \mu \)m from the surface in an effort to minimize the wall effects. The random fluctuations of the microsphere were detected on the QPD position sensor and then curve-fit using the mathematical model of the system according to Eq. (6) (see Fig. 5). A second pole was detected at approximately 11 kHz, and was accounted for during curve-fitting. This pole was due to the dynamic effect of sensing an IR laser with a silicon photodiode [29,30].

The position of the laser, \( u \), is determined by making a model to estimate the actuator’s dynamics. This was necessary due to the lack of adequate position sensors on the FSM. The actuator dynamics were determined by centering the trapping laser on a microsphere that was adhered to the slide surface and mimicking the appearance of a trapped microsphere. This effectively removed the dynamics of a free bead and left only the dynamics of the actuator with the laser and QPD as the position sensor. A broadband noise source was then used to excite the system and produce the actuator’s transfer function. This model was then curve-fit with a fourth-order model (see Fig. 6):

\[
G_A(s) = g \frac{\left(\frac{s + z_1}{s + p_1}\right)}{(s + p_1)(s + p_2)(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{25}
\]
which was the simplest model that closely matched the magnitude and phase of the transfer function. It is important to note that $\hat{G}_A$ is a model of $G_A$, and is subject to errors, which could be due to poor system identification, nonlinearities, such as using a circular beam on a QPD, high-frequency dynamics, and variations of the plant with respect to time, among others.

The designed controller is a PI controller, type I with zero steady-state errors when tracking step inputs. To increase bandwidth and stability margins, a notch filter was included in the controller in order to decrease the effect of the actuator dynamics. This notch filter is centered at the actuator’s natural frequency, decreasing the actuator’s resonant peak seen in the loop-gain while maintaining a unity magnitude elsewhere. The resulting loop-gain has a gain margin and phase margin of 12 dB and 77°, respectively. The loop gain, sensitivity, and complementary sensitivity of the system can be seen in Fig. 7. The achieved bandwidth is $\sim 150$ Hz, as seen in Fig. 7. The sensitivity, $S$, dictates closed-loop disturbance rejection. While the controlled system suppresses disturbances (e.g., from Brownian fluctuations) at low frequencies, the bead’s motion tracks disturbances at high frequencies. Near the crossover frequency, disturbances are amplified as seen in the bead’s motion. It should be noted that this is a consequence of any feedback control scheme as described by the Bode sensitivity theorem. Such effects can be reduced by using a less aggressive controller (lower gain and lower bandwidth) but come with the cost of lower performance.

5. Discussion

A comparison of the open-loop and closed-loop response is shown in Fig. 8. As expected the closed-loop system rejects disturbances at lower frequencies, but amplifies fluctuations near the crossover frequency. With control, the overall rms displacement of the bead is 7% higher than without control—again, such a change is a consequence of any control system, the degree depending upon the aggressiveness of the controller.

Clearly, the effect of control could have a deleterious effect on the desired operation and use of an optical trap if it is not designed properly. While the controlled system, as presented in this work, has an $\sim 10\%$ increase in the rms. displacement, the low-frequency response of the system performs much better than its open-loop counterpart. It should be observed, however, that the consequences of the Bode sensitivity theorem, the amplification near the crossover frequency, do not necessarily imply negative or unusable results in, for example, single-molecule experiments. Provided the bandwidth of the experiment is below the crossover frequency, then such effects will not appear in the measured signal.

Furthermore, experimental data is often low-pass filtered in order to minimize the effects of Brownian motion. The filtering of data has been used in the past when making biological position and force measurements. Because the relevant signal is at lower frequencies, many experimental measurements do not require a high bandwidth of operation. When measuring the step size of kinesin, Svoboda et al. [6] filtered the data at 15 Hz, while Molloy et al. [31] studied the mechanics of myosin which has rate constants on the order of 20 Hz. Many other biological applications are conducted at video camera rates, which are on the order of 30 Hz [4,32] depending on the camera. Visscher et al. [12] created the first force clamp in which the system was sampled at 20 kHz, and the data was filtered at 190–210 Hz. This is on the order of the bandwidth of the system presented herein.

We digitally filtered the measured data and analyzed the effect on the rms displacement. By applying a fifth-order low-pass Butterworth filter with $\omega_c = 100$ Hz to Fig. 8, the closed-loop system has a 20% decrease in rms displacement compared to the open-loop system. By lowering the filter’s cutoff frequency to 50 Hz, the closed-loop rms displacement is half the open-loop rms displacement (see Fig. 9). The lower the cutoff frequency, the larger the increase in performance. As can be seen, the closed-loop system successfully rejects low-frequency disturbances.
Optical traps use light to exert forces on microscopic objects, resulting from the change in momentum of the light as it is refracted through an object. The magnitude of these forces is small, and is a function of the position of objects held in the trap, characterized by the trap stiffness. A feedback control scheme using a QPD position sensor and an FSM actuator was successfully implemented to control the position of the trapped object. While the controller amplified some of the high frequency content, the filtered closed-loop system clearly outperformed its filtered open-loop signal.

### 6. Summary and Conclusions

This system can also be easily converted to a force clamp optical trap. The only modification to the system is to change the reference position, \( r \) (see Fig. 2), from zero to a nonzero number. By multiplying this constant displacement by the calibrated trapping stiffness \( k \), a constant force is applied. Since the controller is implemented digitally, it is easily configurable to the specific experiment being conducted.

This paper also discussed some of the fundamental limits to performance for feedback control of optical traps using an ideal, nominal plant, \( G(s) = \Omega / (s + \Omega) \), and Bode’s ideal loop-gain, \( L(s) = \omega_s / s \), where \( \Omega \) is the trap bandwidth and \( \omega_s \) is the control bandwidth. This loop-gain achieves disturbance rejection within the bandwidth \( (0, \omega_s) \) and perfect disturbance rejection at zero frequency, tracking a step response with zero steady-state error. The resulting controller is \( K(s) = \alpha (s + \Omega) / s \) and performance is determined by the proportional gain \( \alpha = \omega_s / \Omega \). Larger gains result in a smaller mean-square position response, but are limited by the allowable motion of the trap. In particular, the motion of the trapped object cannot exceed the bounds of the trap.

### Appendix A: Analytic Constraints

#### Theorem 1 (Bode sensitivity theorem). Let \( L(s) \) be the loop-gain such that \( sL(s) \to 0 \) as \( s \to \infty \). If the sensitivity \( S = (1 + L(s))^{-1} \) is stable, then

\[
\int_0^\infty \ln|S(j\omega)|\,d\omega = \pi \sum p_i,
\]

where \( p_i \) are the right-half plane poles of \( L(s) \), \( \text{Re} \, p_i > 0 \). If \( L(s) \) has no right-half plane poles then the right hand side is zero.

The consequence of the Bode sensitivity theorem is that the log of the sensitivity must be conserved; that is, good disturbance rejection in one frequency range such that \( |S(j\omega)| < 1 \) must be paid for with \( |S(j\omega)| > 1 \) elsewhere. Some mistakenly assume that rejection over a finite frequency range can be accounted for by making the log sensitivity arbitrarily small over an infinite frequency range. However, all systems have a finite bandwidth and there are practical limits on the gain of amplifiers, etc., requiring the loop-gain to be small above some frequency \( \Omega \). This frequency is often the practical bandwidth of the system. A further consequence is that good disturbance rejection cannot be achieved even over this bandwidth; the penalty can be quite severe as the disturbance control bandwidth approaches the system's open-loop bandwidth as shown in the following theorem.

#### Theorem 2 For a system with an open-loop bandwidth \( \Omega \) and disturbance rejection such that the sensitivity is attenuated by a factor \( M \), \( |S(j\omega)| < M^{-1} < 1 \) over a bandwidth \( \omega_1 < \Omega \), a lower bound for the peak magnitude of the sensitivity is

\[
\|S\|_\infty > M^{\alpha/(1-\alpha)} = M^{\omega_1/(1-\alpha)},
\]

where \( \alpha = \omega_1 / \Omega \). Thus, good disturbance rejection can only be achieved over a significant portion of the available bandwidth at the price of a large \( \|S\|_\infty \).

**Proof.** Assume a stable loop-gain, and that above the bandwidth \( \Omega \), \( |L(j\omega)| \ll 1 \). The Bode sensitivity integral can be broken up as

\[
\int_0^\Omega \ln|S(j\omega)|\,d\omega + \int_{\omega_1}^\infty \ln|S(j\omega)|\,d\omega = 0.
\]

Since good disturbance rejection is desired below the frequency \( \omega_1 \), then \( |S(j\omega)| < M^{-1} < 1 \) in that range and
\[
(\Omega - \omega_1)\ln|S|_\Omega > \int_{\omega_1} \ln|S(i\omega)|d\omega > \omega_1 \ln M. \quad (4)
\]

The result follows directly.

References