ME536/PHY556
ASSIGNMENT 5

Due in class 03/10/2015

Note: All readings are from class notes

Reading: class notes of lectures on RT with ablation

Problems:

Re-derive step by step the dispersion relation for the ablative RT instability presented in class. Make use of lecture notes as needed. Remember that the equilibrium relations are:

\[ \rho U = \rho_h U_h = \rho_f U_f = \text{const}, \]
\[ p = R \rho T \approx \text{const} \Rightarrow \rho_h T_h \approx \rho_f T_f, \]
\[ \frac{d\xi}{dz} = \frac{\xi^{y+1}(1 - \xi)}{L_0}, \quad \xi = \frac{\rho}{\rho_h}, \]
\[ L_0 = -\frac{k(T_h)}{\rho_h U_h c_p}, \]
\[ U_h = -V_a < 0, \]
\[ \tilde{g} = -g \tilde{e}_z, \quad g > 0. \]

(a) Solve the linearized equations of motion in the heavy fluid. Assume that \( \nabla \cdot \tilde{v} = 0 \). Show that:

\[ \tilde{\rho}_h = 0, \]
\[ \tilde{v}_z^h = \tilde{u}_h e^{-kz}, \]
\[ \tilde{v}_x^h = -i \tilde{u}_h e^{-kz}, \]
\[ \tilde{p}_h = \frac{\rho_h}{k} (\gamma - k U_h) \tilde{u}_h e^{-kz}. \]
(b) Solve the linearized equations in the light fluid. The model equations in the light fluid are:

\[ \rho (\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v}) = -\nabla p \]
\[ \nabla \cdot \vec{v} = 0 \]
\[ \nabla^2 \Phi = 0 \leftarrow \text{this comes from energy equations for } \mathbf{v} \gg 1 \]
(i.e.) large thermal conduction in the light fluid
\[ \rho T = \text{const} \]
\[ \phi = \left( \frac{T}{T_h} \right)^{\nu+1} \left( \frac{\mu}{\rho} \right)^{\nu+1} \frac{1}{\nu+1} \]

Notice that in the derivations of the model equations we have assumed:

\[ \nabla \cdot \vec{v} = 0, \quad \mathbf{v} \gg 1, \quad \tilde{\rho}, g \ll \nabla \tilde{p}, \quad \frac{\tilde{p}}{p} \ll \frac{\tilde{\rho}}{\rho} \]

Linearize those equations, solve them and show that:

\[ \tilde{v}_z = \tilde{u}, e^{kz} + \tilde{b} e^{-\nu/\mathbf{U}_k} \]
\[ \tilde{v}_x = i \left( \tilde{u}, e^{kz} - \frac{\gamma}{k \mathbf{U}_k} \tilde{b} e^{-\nu/\mathbf{U}_k} \right) \]
\[ \tilde{p}_x = -\frac{\tilde{u}}{k}, \rho, (\gamma + k \mathbf{U}_k) e^{kz} \]
\[ \tilde{\phi} = \phi e^{kz} \]
\[ \tilde{\rho}_x = -\rho, \phi \left( \frac{\rho}{\rho_h} \right)^{\nu+1} e^{kz} \]

(c) From the light fluid side, treat the ablation front as isothermal \( \rightarrow \Phi(t, x, z = \eta(x, t)) = \text{const} \)
Linearize this equation and show that

\[ \dot{\phi} = -\left[ \frac{d\Phi}{dz} \right]_{z=0} \tilde{\eta} \] where the derivative is from the light fluid side.
(d) Using the equilibrium equation
\[ \frac{d\xi}{dz} = \frac{\xi_0^{\gamma+1}(1-\xi)}{L_0} \]
(where \( \xi = \frac{\rho}{\rho_h} \)) show that:

\[ \Phi = \frac{\tilde{n}}{L_0} \frac{1}{\xi_0}(1-\xi), \quad \left[ \frac{d^2\Phi}{dz^2} \right]_0 = \frac{\xi_0^{\gamma+1}(1-\xi)}{L_0^2} \]

(e) Derive the jump conditions and linearize them. Show that:

\[ \gamma_0(\rho_h - \rho_i) = \rho_h \tilde{v}_z + \rho_h U_h - \rho_i \tilde{v}_z' - \rho_i U_i \quad \left\langle \text{from mass conservation} \right\rangle \\
\tilde{v}_z - \tilde{v}_z' + i k \tilde{\eta}(U_h - U_i) = 0 \quad \left\langle \text{from tangential momentum} \right\rangle \\
\tilde{p}_h - \tilde{p}_i + \rho_h U_h^2 - \rho_i U_i^2 + 2 \rho_h U_h (\tilde{v}_z - \tilde{v}_z') - (\rho_h - \rho_i) g \tilde{\eta} = 0 \quad \left\langle \text{from normal momentum} \right\rangle \\
\frac{\Gamma p}{\Gamma - 1} (\tilde{v}_z - \tilde{v}_z') = \kappa_h T_h \left[ \left( \frac{d\Phi}{dz} \right)_{z=\eta'} - \left( \frac{d\Phi}{dz} \right)_{z=-\eta} \right] \quad \left\langle \text{from energy} \right\rangle \\

(f) Using the heavy and light fluid solutions, and the equilibrium equation show that

\[ \left[ \left( \frac{d\Phi}{dz} \right)_{z=\eta'} - \left( \frac{d\Phi}{dz} \right)_{z=-\eta} \right] = - \frac{k \tilde{\eta}}{L_0 \xi_0} \frac{1}{\xi_0} \left( 1 + \frac{\xi_0^{\gamma+1}}{kL_0} \right) \]

Then use the sharp boundary limit \( L_0 = \lim_{\nu \to \infty} \nu \frac{\xi_0^{\gamma+1}}{k} \to 0 \) to show that the ratio into the bracket is small. Then use the physical definition of \( L_0 \) on page 1 and rewrite the energy jump condition as

\[ \tilde{v}_z - \tilde{v}_z' = k U_h \tilde{\eta}(1-\xi) \]

(g) Neglect the term \( \tilde{p}_i \) in the jump conditions from mass and normal momentum. Using the energy jump condition into the mass conservation jump condition show that

\[ \tilde{\eta}(\gamma - k U_h) = \tilde{v}_z \quad \left\langle \text{interface equation} \right\rangle \]
Collect all the final forms of the jump conditions:

\[
\begin{align*}
\tilde{\eta}(\gamma - k U_h) &= \tilde{v}_z^h \\
\tilde{v}_z^h - \tilde{v}_z^f &= k U_f \tilde{\eta}(1 - \xi) \\
\tilde{v}_x^h - \tilde{v}_x^f + i k \tilde{\eta}(U_h - U_f) &= 0 \\
\tilde{p}_h - \tilde{p}_f + 2 \rho_h U_h (\tilde{v}_z^h - \tilde{v}_z^f) - (\rho_h - \rho_f) g \tilde{\eta} &= 0
\end{align*}
\]

Substitute the solutions from the heavy and light fluid and set the determinant to zero (i.e. find the dispersion relation for the growth rate). Show that the growth rate can be written as:

\[
\gamma = \sqrt{A k g - A^2 k^2 U_h U_f - (1 + A) k |U_h|}
\]

where:

- \( U_h = -|V_a| \) represents the ablation velocity
- \( U_f = \frac{V_a}{\xi} \) represents the blow-off or light fluid velocity
- \( A = \frac{\rho_h - \rho_f}{\rho_h + \rho_f} \) represents the Atwood number
- \( g \) is the acceleration
- \( k \) is the mode wave number

Verify your assumptions and derive the limits under which your assumptions are correct.