Kelvin-Helmholtz Instability

The K-H instability is driven by the shear in velocity (i.e., velocity gradient).

Consider two incompressible fluids flowing at different velocities $U_1, U_2$.

\[ \frac{d_1}{U_1} \quad \frac{d_2}{U_2} \]

The fluid motion is governed by the standard incompressible fluid equations.

\[ \rho = \text{constant} \quad \nabla \cdot \mathbf{V} = 0 \quad (1) \]

\[ \rho \left( \mathbf{D} + \mathbf{v} \cdot \nabla \right) \mathbf{V} = -\nabla p \quad (2) \]

It is convenient to use the frame of reference moving with the average velocity.

\[ \langle \mathbf{U} \rangle = \frac{U_1 + U_2}{2} \]
In this frame of reference, fluid 1 moves with velocity \( U_1 - \langle U \rangle = \frac{U_1 - U_2}{2} = U \)

and fluid 2 with velocity \( U_2 - \langle U \rangle = \frac{U_2 - U_1}{2} = -U \)

\[
\begin{array}{c}
U \\
\downarrow \\
\rightarrow \text{\ 1} \\
\hline
\text{\ 2} \\
\downarrow \quad \quad \\
-\ U
\end{array}
\]

Equilibrium: since \( U \) is constant then Eq. (2) yields \( \nabla p = 0 \Rightarrow p = \text{constant} \).

Linear Stability:

3. \( \nabla \cdot \vec{V} = 0 \Rightarrow \vec{V}_x = \frac{i}{k} \partial_z \vec{V}_z \)

4. \( p (\rho + iKV) \vec{V}_x = -iK \rho \) \( x \)-momentum

5. \( p (\rho + iKV) \vec{V}_z = -\partial_z \rho \) \( z \)-momentum

here \( \vec{V} = U \) in fluid 1 and \( \rho = \rho_1 \)

\( \vec{V} = -U \) in fluid 2 and \( \rho = \rho_2 \)
Rewrite (4) as \( \tilde{p} = \frac{i}{k} \int (k + i k V) \tilde{\phi}_x \).

Substitute into (5) and use (3)

\[
p(k + i k V) (k^2 - \partial_z^2) \tilde{\gamma}_z = 0
\]

\[
\tilde{\gamma}_z = \hat{u}_z e^{-k z} \quad \text{in fluid 1}
\]

\[
\tilde{\gamma}_z = \hat{u}_z e^{k z} \quad \text{in fluid 2}
\]

**JUMP CONDITIONS.**

1. \( l_z = \eta(x, t) \)

Use the standard change of coordinates:

\( z = z - \eta(x, t) \), and find (see Ablative RT lecture notes)

**x-momentum**

\[
-\dot{\gamma} \rho \dot{u}_x + \rho \dot{u}_z u_x - \partial_x \left( \rho + \rho u_x^2 \right) = 0
\]

**z-momentum**

\[
-\dot{\gamma} \rho \dot{u}_z + \rho + \rho u_z^2 - \partial_z \left( \rho u_x \dot{u}_z \right) = 0
\]

**mass conservation**

\[
-\dot{\gamma} \rho + \rho \dot{u}_z - \partial_z \rho u_x = 0
\]
\text{\textbf{L\text{\textasciitilde}m\text{\textasciitilde}ern\text{\textasciitilde}bc\text{\textasciitilde}.}}

\begin{align*}
X - \text{mom} & = \hat{\tilde{\gamma}} (p_1 V_1 - p_2 V_2) + p_1 \tilde{\varpi}_{21} V_1 - p_2 \tilde{\varpi}_{22} V_2 - i K \tilde{\gamma} (p_1 V_1 - p_2 V_2) = 0 + \\
Z - \text{mom} & = \tilde{p}_1 = \tilde{p}_2
\end{align*}

\begin{align*}
m_{21} & = - \hat{\tilde{\gamma}} (p_1 - p_2) + p_1 \tilde{\varpi}_{21} - p_2 \tilde{\varpi}_{22} - i K \tilde{\gamma} (p_1 V_1 - p_2 V_2) = 0 \quad 9)
\end{align*}

\text{Rewrite} \ X - \text{mom} \Rightarrow \begin{align*}
& [ (\chi + i K V_1) \tilde{\gamma} - \tilde{\varpi}_{21} ] \ p_1 V_1 = [ (\chi + i K V_2) \tilde{\gamma} - \tilde{\varpi}_{22} ] \ p_2 V_2. \quad 10)
\end{align*}

\begin{align*}
m_{22} & \rightarrow \begin{align*}
& [ (\chi + i K V_1) \tilde{\gamma} - \tilde{\varpi}_{21} ] \ p_1 = [ (\chi + i K V_2) \tilde{\gamma} - \tilde{\varpi}_{22} ] \ p_2. \quad 11)
\end{align*}
\end{align*}

Since \( V_1 \neq V_2 \) those two equation can only be satisfied if

\( (\chi + i K V_1) \tilde{\gamma} = \tilde{\varpi}_{21} \quad \text{and} \quad (\chi + i K V_2) \tilde{\gamma} = \tilde{\varpi}_{22}. \quad 12) \)

Using the expression for \( \tilde{p} \) on page 3 and Eq. 3 for \( \tilde{V}_x \) one finds

\begin{align*}
\begin{bmatrix} \tilde{p}_1 = \tilde{p}_2 \end{bmatrix}_{2 \times 2} \Rightarrow \begin{Bmatrix}
\begin{bmatrix} \tilde{p}_1 (\chi + i K V_1) (\tilde{\varpi}_{21}) \end{bmatrix}_{1 \times 1} = \begin{bmatrix} \tilde{p}_2 (\chi + i K V_2) (\tilde{\varpi}_{22}) \end{bmatrix}_{1 \times 1}
\end{Bmatrix}
\end{align*}

\text{\textbf{Fig.}}
Use the fluid 1, 2 solutions\[\tilde{u}_1 = U_1 e^{-kz}\]
\[\tilde{u}_2 = U_2 e^{kz}\]
and rewrite the jump conditions \((12), (13)\) as:
\[
\begin{align*}
\left(\gamma + kV_1\right)\tilde{\eta} &= \tilde{u}_1 \\
\left(\gamma + kV_2\right)\tilde{\eta} &= \tilde{u}_2 \\
-k_1 (\gamma + kV_1) \tilde{u}_1 &= k_2 (\gamma + kV_2) \tilde{u}_2.
\end{align*}
\]

Eliminate \(\tilde{u}_1, \tilde{u}_2\) from bottom equations.

\[
P_1 (\gamma + kV_1) \tilde{\eta} + P_2 (\gamma + kV_2) \tilde{\eta} = 0
\]

\[
P_1 (\gamma + kV_1)^2 + P_2 (\gamma + kV_2)^2 = 0
\]

**Dispersion Relation**

\[
(P_1 + P_2) \xi^2 + 2i k \gamma \left[ P_1 V_1 + P_2 V_2 \right] - k^2 \left( P_1 V_1^2 + P_2 V_2^2 \right) = 0.
\]

Remember that in the frame of reference moving with the average velocity \(V_1 = U = (U_1 + U_2)/2\)
\(V_2 = -U = - (U_1 - U_2)/2\).
Then the dispersion relation becomes

\[(p_1 + p_2)^2 + 2iKU \cdot (p_1 - p_2) - k^2 U^2(p_1 + p_2) = 0\].

\[\gamma^2 + 2iKU \cdot A - k^2 U^2 = 0\]

\[A = \frac{p_1 - p_2}{p_1 + p_2}\].

Solve for \(\gamma\):

\[\gamma = -iKUA \pm \sqrt{-k^2 U^2 A^2 + k^2 U^2}\].

\[\gamma = -iKUA \pm |KU| \sqrt{1 - A^2} = -iKUA \pm |KU| \sqrt{\frac{4p_1 p_2}{(p_1 + p_2)^2}}\].

Observe that the + root has a positive real part and therefore is unstable (K-H instability).

\[\gamma = -iKUA + |KU| \sqrt{\frac{4p_1 p_2}{(p_1 + p_2)^2}}\].

growth rate.

\[\gamma = \text{oscillatory component}\].