This final exam is a relatively simple research problem. To the best of my knowledge, this problem has not yet been solved and you may or may not find a final solution for it. What really matters is that you demonstrate your ability to carry out new research. This final exam is about the nonlinear Rayleigh-Taylor instability of classical fluids. In class, we have investigated the so-called Layzer model (Layzer, 1955) for an infinite medium incompressible fluid with constant density. There are two effects (not included into the Layzer model). One effect is the one of finite thickness of the fluid and the other is the one of time varying bulk density.

For the final exam, you will modify the Layzer’s model to include the effect of non-constant (in time) heavy fluid density while keeping the heavy fluid infinite. The density is still spatially uniform.

Using the mass conservation equation, assuming that the heavy fluid density is an assigned function of time \( \rho = \rho(t) \) and assuming that the velocity can be described by a potential function \( v = -\nabla \Phi \) show that the function \( \Phi \) satisfies the Poisson’s equation

\[
\nabla^2 \Phi = \frac{\dot{\rho}}{\rho}
\]

Using the same procedure as in the notes, show that the most general solution of this equation is

\[
\Phi = \Phi_1(t)e^{-kz} \cos(kx) + \Phi_2(t)z + \frac{\dot{\rho}}{\rho} \frac{z^2}{2}
\]

where \( \Phi_1 \) and \( \Phi_2 \) are two arbitrary functions of time. Notice that the last term represents the equilibrium velocity required to change the density in time. The density is assigned at all time.

Modify the Layzer’s model for infinite medium to include the new terms in the above equations. Find an expression for the asymptotic value of the bubble velocity. If you have difficulty to find an answer for arbitrary \( \rho(t) \), then use \( \rho(t) = \rho(0)e^{\alpha t} \) which turns the right-hand-side of the Poisson’s equation into a constant. Use the same expansion about the bubble tip as in the notes and, as in the notes, take the limit of vanishing light fluid density.