Asymptotic Scaling Laws for Imploding Thin Fluid Shells

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Scaling laws governing implosions of thin shells in converging flows are established by analyzing the implosion trajectories in the \((A,M)\) parametric plane, where \(A\) is the in-flight aspect ratio, and \(M\) is the implosion Mach number. Three asymptotic branches, corresponding to three implosion phases, are identified for each trajectory in the limit of \(A,M \gg 1\). It is shown that there exists a critical value \(\gamma_c = 1 + 2/\nu\) (\(\nu = 1, 2\) for, respectively, cylindrical and spherical flows) of the adiabatic index \(\gamma\), which separates two qualitatively different patterns of the density buildup in the last phase of implosion. The scaling of the stagnation density \(\rho_\text{s}\) and pressure \(P_\text{s}\) with the peak value \(M_0\) of the Mach number is obtained.

Analytically derived scaling laws are validated by numerical simulations.

Once a specific implosion strategy is chosen and fixed, the entire multitude of all possible states of imploding shells may be considered as a five-parameter family of snapshot spatial profiles for the density \(\rho(t,r)\), velocity \(u(t,r)\), and pressure \(p(t,r)\). We choose the peak density \(D = D(t) = \max_\rho \rho(t,r) = \rho(t,R)\), the corresponding pressure \(P(t) = p(t,R)\), the implosion velocity \(U(t) = -u(t,R)\), the shell radius \(R(t)\), and its effective thickness

\[
h = h(t) = \frac{1}{D} \int_0^\infty \rho(t,r) \, dr
\]

to be such parameters.

Optimal implosion strategy for the highest degree of final compression, as described in detail in Ref. [4], is accomplished by (i) setting a cold shell in motion by a carefully tailored pressure pulse, which generates a minimum amount (if any) of entropy, followed by (ii) the phase of adiabatic acceleration to a maximum implosion velocity by the peak boundary pressure \(P_0\). Here we omit the first phase—which is absolutely insignificant for the scaling laws in question—and start with an isentropic density profile across a motionless shell,

\[
\rho(0,r) = D_0 \left( \frac{r - r_0}{R_0 - r_0} \right)^{1/(\gamma - 1)},
\]

which corresponds to a uniform acceleration of all fluid elements (a UA, i.e., uniformly accelerated, profile) by a fixed pressure \(P_0\) at the outer boundary; \(\gamma\) is the adiabatic index. The interior of the shell at \(0 < r < r_0\) is void. The initial effective shell thickness is \(h_0 = (1 - \gamma^{-1})(R_0 - r_0)\). Under such drive conditions no shocks pass through the shell until the very moment of void closure, and the entropy parameter \(\alpha = P/p\gamma = \alpha_0 = P_0/D_0^\gamma\) remains constant in space and time.

From the five-dimensional parameters \(P, D, U, R,\) and \(h\), which define the state of a shell in flight, only two dimensionless combinations can be constructed, namely,
the in-flight aspect ratio \(A = R/h\), and the Mach number \(M = U/C\). [To simplify the formulas, we define the Mach number with respect to the isothermal speed of sound \(C = (P/D)^{1/2}\) which, hence, each implosion is represented by a trajectory in the \((A, M)\) parametric plane. All possible implosions (within a fixed implosion strategy) are given by a one-parameter family of such trajectories. As a dimensionless parameter, identifying an individual implosion trajectory, either the initial aspect ratio \(A_0 = R_0/h_0\) or the maximum Mach number \(M_0\) [see Eq. (7) below] can be chosen. Clearly, such dimensionless quantities as the ratios between the stagnation and the initial pressures and densities are uniquely determined by the value of \(A_0\) (or \(M_0\))—in combination, of course, with the \(\gamma \) and \(\nu\) values, where \(\nu = 0, 1, 2\) corresponds, respectively, to planar, cylindrical, and spherical flows.

We base our analysis on the equations of mass balance,

\[
m = DhR^2 = D_0h_0R_0^2 = \text{const},
\]

energy balance,

\[
P_0(R^{\nu+1} - R^{\nu+1}) = \frac{1}{2} mU^2, \tag{4}
\]

and a comparison between two relevant time scales

\[
t_h = \frac{h}{C} \quad \text{and} \quad t_{im} = \frac{R}{U}, \tag{5}
\]

where \(t_h\) is the time scale of hydrodynamic relaxation of the density (pressure) profile across the shell, and \(t_{im}\) is the implosion time. Equations (3) and (4) apply to thin \((A \gg 1)\) shells in the limit of \(M \gg 1\). For simplicity, we have omitted the unimportant geometrical factor \(4\pi (2\pi)\) for spherical (cylindrical) shells.

Relation between the time scales \(t_h\) and \(t_{im}\) divides the \((A, M)\) plane into two domains. In regions \(A\) and \(B\), where the two initial phases of implosion occur (see Fig. 1), the ratio \(t_h/t_{im} = M/A \ll 1\). This means that the density distribution across the shell relaxes to the equilibrium profile, corresponding to the adopted boundary condition, much faster than the effects of flow convergence distort it. As a result, the density profile preserves the UA shape corresponding to the fixed boundary pressure \(P_0\), with \(D = D_0\) being constant and the areal density varying as \(\int p \, dr = D_0h \propto R^{-\nu}\). This condition, combined with Eqs. (3) and (4), leads to the following analytical expression for the implosion trajectories:

\[
M = \left[\frac{2}{\nu + 1} (A_0 - A)\right]^{1/2} = \left(M_0^2 - \frac{2}{\nu + 1} A\right)^{1/2}. \tag{6}
\]

Here we have introduced the maximum Mach number

\[
M_0 = \left(\frac{2}{\nu + 1} A\right)^{1/2}. \tag{7}
\]

The divide line between the phase \(A\) of initial acceleration and the phase \(B\) of “relaxed” converging flow is given by

\[
\text{line } b: \quad M = \left(\frac{2}{\nu + 1} A\right)^{1/2}. \tag{8}
\]

Figure 1 demonstrates a perfect agreement between Eq. (6) and the results of numerical simulations in regions \(A\) and \(B\). Simulations have been done with the 1D hydrodynamics code DEIRA [5], based on a Lagrangian scheme with a tensor artificial viscosity. The DEIRA code has been designed and extensively tested for simulating ICF fusion capsules. For the present problem, the code has been modified to treat a central void cavity. The initial state was assigned as an isentropic UA profile (2) at rest, and a fixed pressure \(p = P_0\) was used as an outer boundary condition. A zero pressure before the void closure and a zero velocity after the void closure were used as a boundary condition at the inner shell edge. Typically, 150 mesh cells across the shell thickness were used.

Examination of the implosion dynamics in region \(C\), where \(t_h \gg t_{im}\), reveals that there exists a critical value of the adiabatic index,

\[
\gamma_{cr} = 1 + \frac{2}{\nu}, \tag{9}
\]

which separates two qualitatively different implosion patterns. For an imploding state in region \(C\), the density buildup due to flow convergence occurs much faster than it can relax to the pertinent boundary condition. Then, one can tentatively assume that the shell thickness \(h(t)\) becomes asymptotically “frozen” at a certain constant value \(h_1\) as the shell radius \(R \rightarrow 0\). Under this assumption the peak density increases as \(D \propto R^{-\nu}\), and the large ratio

\[
\frac{t_h}{t_{im}} = \frac{M}{A} \propto D^{(1-\gamma)/2} R^{-1} \propto R^{\nu(\gamma-1)/2-1} \tag{10}
\]

becomes even larger as \(R \rightarrow 0\) for \(\gamma < \gamma_{cr}\) [the implosion velocity \(U\) saturates at its limiting value, given by Eq. (4) for \(R \ll R_0\)]. As a consequence, the assumption of a frozen shell thickness \(h\) becomes better and better justified, and the implosion trajectory, once in region \(C\), tends to

FIG. 1. Implosion trajectories for (a) \(\gamma < \gamma_{cr}\) and (b) \(\gamma > \gamma_{cr}\). Results of numerical simulations (thin solid) are compared with the asymptotic expression (6) for regions \(A\) and \(B\) (dashed), and Eq. (12) for region \(C\) (dash-dotted). Lines \(b\) and \(c\) separate three successive phases (\(A, B,\) and \(C\)) of implosion.
penetrate deeper in this region, away from the divide line \( c \) where \( t_h \approx t_{im} \), and approaches asymptotically the power law
\[
M \propto C^{-1} \propto D^{(1-\gamma)/2} \propto A^{\gamma/2}.
\] (11)

The latter is clearly illustrated in Fig. 1a with numerically calculated implosion trajectories.

For \( \gamma > \gamma_{cr} \), on the opposite, the ratio \( t_h/t_{im} \), once initially large, decreases with \( R \to 0 \), and an implosion trajectory, once in region \( C \), is pushed out back to region \( B \). The latter means that, for a fixed boundary pressure, implosion trajectories never actually cross to region \( C \). Rather, having approached the divide line \( M \propto A \) from region \( B \), they turn away from the asymptotic law (6) and follow the line \( M \propto A \), corresponding to \( t_h \approx t_{im} \). The combined effect of the density growth due to flow convergence and the competing relaxation due to hydrodynamic expansion, both of which act on the same time scale \( t_h \approx t_{im} \), results in that finally all the implosion trajectories with different \( M_0 \) develop the same asymptotic density profile and converge to a single (for given \( \nu \) and \( \gamma > \gamma_{cr} \)) path (an attractor) in the \((A,M)\) plane — as it is clearly seen in Fig. 1b.

The above considerations, combined with Eqs. (3) and (4), lead to the following asymptotic expressions for the implosion trajectories in phase \( C \),
\[
M = \begin{cases} 
M_0^{1-\nu/(\gamma-1)/2} (K_{c0}A)^{\nu/(\gamma-1)/2}, & \gamma < \gamma_{cr}, \\
K_{c0}A, & \gamma > \gamma_{cr}.
\end{cases}
\] (12)

Here we have introduced two unknown coefficients, namely, \( K_{c0} \), which defines the divide line,
\[
\text{line } c: M = K_{c0}A,
\] (13)

between regions \( B \) and \( C \) in the case of \( \gamma < \gamma_{cr} \), and the slope \( K_{cY} \) of the limiting implosion path for \( \gamma > \gamma_{cr} \). The dependence on \( M_0 \) in Eq. (12) for \( \gamma < \gamma_{cr} \) is recovered by observing that lines (11) and (13) both cross the \( M = M_0 \) line at the same point. From numerical simulations we infer the value of \( K_{c0} = 10.0 \pm 0.5 \), which shows practically no dependence on \( \nu \) and \( \gamma \). The coefficient \( K_{cY} \) can be calculated analytically (the details are to be published elsewhere),
\[
K_{cY} = \frac{\gamma + 1}{\gamma - 1} \frac{2\pi \gamma}{\Gamma(\frac{\gamma}{\gamma-1})}\left[\frac{\nu(\gamma - 1) - 2}{\nu + 1}\right]^{1/2}
\times \frac{1}{\Gamma\left(\frac{\gamma}{\gamma-1}\right)},
\] (14)

where \( \Gamma(x) \) is Euler’s gamma function. The asymptotic law of density growth in phase \( C \)—the phase of compression by flow convergence—is given by
\[
D \propto \begin{cases} 
R^{-\nu}, & \gamma < \gamma_{cr}, \\
R^{-2(\nu+1)/(\nu+1)}, & \gamma > \gamma_{cr}.
\end{cases}
\] (15)

Implosion of a shell ends when the central void closes and a return shock stops the infalling material. This occurs when the aspect ratio \( A \) falls below a certain value \( A_{cr} \), which, in general, depends on \( \nu \) and \( \gamma \). Equation (12) tells us that the amplitude of the return shock, determined by the corresponding Mach number,
\[
M_1 \propto \frac{M_0^{1-\nu/(\gamma-1)/2}}{K_{c0}A}, \quad \gamma < \gamma_{cr},
\] (16)

becomes infinitely large in the limit of \( M_0 \to \infty \) for \( \gamma < \gamma_{cr} \), and approaches a finite value for \( \gamma > \gamma_{cr} \). Hence, the return shock compresses the infalling matter by a limiting factor \( (\gamma + 1)/(\gamma - 1) \) for \( \gamma < \gamma_{cr} \), and by a smaller finite factor for \( \gamma > \gamma_{cr} \). As a result, we obtain the following scaling law:
\[
\frac{\rho_s}{\rho_0} \propto \frac{M_0}{M_1}^{2/(\gamma-1)} \propto \begin{cases} 
M_0^{\gamma_{cr}/(\gamma-1)}, & \gamma < \gamma_{cr}, \\
M_0^{2/(\gamma-1)}, & \gamma > \gamma_{cr}.
\end{cases}
\] (17)

for the ratio between the stagnation density \( \rho_s \) and the initial peak density \( \rho_0 \equiv D_0 \). Here \( \rho_s \) is some characteristic density at stagnation behind the return shock, representative of the bulk of the shell mass (because of a singularity occurring in the ideal hydrodynamics at \( r = 0 \) at void closure, the infinite peak density behind the return shock is of no practical interest).

The scaling of the stagnation pressure \( P_s \) is established by extending the energy equation (4),
\[
\frac{P_s V_s}{\gamma - 1} + \frac{\rho_0 R_0^{\nu+1}}{\nu + 1} = \frac{1}{2} m P_0 D_0^2 \frac{M_0^{2/(\gamma-1)}}{M_0^{\gamma_{cr}/(\gamma-1)}},
\] (18)
to the stagnation state, for which we ignore the kinetic energy in comparison with the thermal one. Since the total volume \( V_s \) at stagnation scales as \( V_s \propto m/\rho_s \), we arrive at
\[
\frac{P_s}{P_0} \propto \frac{\rho_s}{\rho_0} M_0^{2/(\gamma-1)} \frac{M_0^{\gamma_{cr}/(\gamma-1)}}{M_0^{2/(\gamma-1)}}, \quad \gamma < \gamma_{cr},
\] (19)

Equations (17) and (19) allow us to relate the values of the adiabat parameter \( \alpha = P/\rho^\gamma \) before and after the stagnation,
\[
\frac{\alpha_s}{\alpha_0} = \left(\frac{\rho_s}{\rho_0}\right)^{1-\gamma} M_0^{2/(\gamma-1)} \propto \begin{cases} 
M_0^{1-\nu/(\gamma-1)/2}, & \gamma < \gamma_{cr}, \\
M_0^{\gamma_{cr}/(\gamma-1)}, & \gamma > \gamma_{cr}.
\end{cases}
\] (20)

By using Eq. (7), one can rewrite Eqs. (17), (19), and (20) in terms of the initial in-flight aspect ratio \( A_0 \).

Scaling laws (17), (19), and (20) are fully confirmed by our numerical 1D hydrodynamic simulations. The results for the \( \alpha_s \) scaling are shown in Fig. 2. The agreement for the \( \rho_s/\rho_0 \) and \( P_s/P_0 \) ratios is equally good. For the case of \( \nu = 2 \), \( \gamma = 5/3 \), which is most relevant to ICF, we calculate
\[
\alpha_s = (0.62 \pm 0.01) A_0 M_0^{2/3}.
\] (21)

Because the specific entropy distribution behind the return shock is not uniform, the numerical coefficient in Eq. (21) depends on the choice of the reference point. In simulations, the values of \( \rho_s \), \( P_s \), and \( \alpha_s \) were taken at the midpoint with respect to the mass coordinate of the shell. Remarkably, the asymptotic relationships (17), (19), and (20), valid in the limit of \( M_0 \to \infty \), become already quite accurate for the values of \( M_0 \) as low as \( M_0 \approx 10 \).
The principal difference between our scaling and that of Ref. [2] is that the pressure scaling has been obtained in Ref. [2] as a power-law dependence on the maximum Mach number \( M_0 \) calculated from the self-similar solution for an intermediate range of \( 2 \leq M_0 \leq 20 \). Our scaling, on the contrary, may be considered as an exact analytic result for the limit of \( M_0 \to \infty \). Quantitatively, the difference at \( \gamma < \gamma_{cr} \) is not very dramatic: for \( \nu = 2, \gamma = 5/3 \) we find \( P_s/P_0 \approx M_0^{4/3} \) instead of \( P_s/P_0 \approx M_0^{4/3} \) in Ref. [2]. If, following Ref. [2], we set up an increasing inward entropy profile and simulate cases with \( M_0 < 10 \), we also observe \( P_s/P_0 \approx M_0^{2.5-3.5} \) [in the limit of \( M_0 \to \infty \), Eqs. (17), (19), and (20) apply to the nonuniform entropy profiles as well]. Although the existence of \( \gamma_{cr} \) is not discussed explicitly in Ref. [2], one should be aware that Eqs. (16) and (17) of Ref. [2] become inapplicable for \( \gamma > \gamma_{cr} \) and sufficiently large values of \( M_0 \).

In conclusion, we have established how the basic laws of ideal hydrodynamics govern the compression of matter and the increase of the stagnation adiabat in thin fluid shells that are imploded in converging geometries by a fixed external pressure. The derived scaling laws may serve as valuable guidelines for general analysis of imploding configurations in ICF, and, in particular, for gaining a deeper insight into the scaling laws for a minimum ignition energy [4]. For the latter case, the relationship (20) provides an important link between the in-flight, \( \alpha_0 \), and stagnation, \( \alpha_s \), adiabat parameters. A direct comparison, however, shows no perfect agreement between the formula

\[
\alpha_s \propto \alpha_0 U_{\text{max}}^{2/3} P_0^{-2/15},
\]

obtained for \( \nu = 2, \gamma = 5/3 \) by substituting \( M_0 = U_{\text{max}}(P_0/\rho_0)^{-1/2} = \alpha_0 P_0^{-3/10} U_{\text{max}} P_0^{-5/15} \) into our asymptotic law (20), and the scaling

\[
\alpha_s \propto \alpha_0 \gamma^{0.75 \pm 0.01} \gamma^{0.44 \pm 0.03} P_0^{-0.21 \pm 0.01},
\]

inferred in Ref. [4] from a large number of fusion capsule simulations. This is not surprising because the fusion capsules have been simulated for a certain intermediate range of Mach numbers, had gas filled central cavities, and included other physical processes (such as heat conduction, alpha-particle heating, etc.) which distort the ideal hydrodynamic profiles used in the present work.

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