Problem 2.5

With \( u \) measured from the static equilibrium position of \( m_1 \) and \( k \), the equation of motion after impact is

\[
(m_1 + m_2) \ddot{u} + ku = m_2 g
\]

(a)

The general solution is

\[
u(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{m_2 g}{k}
\]

(b)

\[
\omega_n = \sqrt{\frac{k}{m_1 + m_2}}
\]

(c)

The initial conditions are

\[
u(0) = 0 \quad \dot{u}(0) = \frac{m_2}{m_1 + m_2} \sqrt{2gh}
\]

(d)

The initial velocity in Eq. (d) was determined by conservation of momentum during impact:

\[
m_2 \dot{u}_2 = (m_1 + m_2) \dot{u}(0)
\]

where

\[
\dot{u}_2 = \sqrt{2gh}
\]

Impose initial conditions to determine \( A \) and \( B \):

\[
u(0) = 0 \Rightarrow A = -\frac{m_2 g}{k}
\]

(e)

\[
\dot{u}(0) = \omega_n B \Rightarrow B = \frac{m_2}{m_1 + m_2} \frac{\sqrt{2gh}}{\omega_n}
\]

(f)

Substituting Eqs. (e) and (f) in Eq. (b) gives

\[
u(t) = \frac{m_2 g}{k} \left(1 - \cos \omega_n t\right) + \frac{\sqrt{2gh}}{\omega_n} \frac{m_2}{m_1 + m_2} \sin \omega_n t
\]
Problem 2.9

Equation of motion:
\[ m \dddot{u} + c \dot{u} + k u = 0 \]  
(a)

Dividing Eq. (a) through by \( m \) gives
\[ \dddot{u} + 2 \zeta \omega_n \dot{u} + \omega_n^2 u = 0 \]  
(b)

where \( \zeta > 1 \).

Assume a solution of the form \( u(t) = e^{\omega t} \). Substituting this solution into Eq. (b) yields
\[ (s^2 + 2 \zeta \omega_n s + \omega_n^2) e^{\omega t} = 0 \]

Because \( e^{\omega t} \) is never zero, the quantity within parentheses must be zero:
\[ s^2 + 2 \zeta \omega_n s + \omega_n^2 = 0 \]

or
\[ s = \frac{-2 \zeta \omega_n \pm \sqrt{(2 \zeta \omega_n)^2 - 4 \omega_n^2}}{2} = \begin{cases} -\zeta \pm \sqrt{\zeta^2 - 1} \omega_n \end{cases} \]

The general solution has the following form:
\[ u(t) = A_1 e^{-\zeta \omega_n t} + A_2 e^{(\sqrt{\zeta^2 - 1} \omega_n)t} \]  
(c)

where the constants \( A_1 \) and \( A_2 \) are to be determined from the initial conditions: \( u(0) \) and \( \dot{u}(0) \).

Evaluate Eq. (c) at \( t = 0 \):

\[ u(0) = A_1 + A_2 \Rightarrow A_1 = u(0) \]

\[ A_2 \]

Differentiating Eq. (c) with respect to \( t \) gives
\[ \ddot{u}(t) = -\zeta \zeta \omega_n \dot{u} + \omega_n \exp\left(-\zeta \zeta \omega_n t\right) \]

Evaluate Eq. (a) at \( t = 0 \):
\[ \dot{u}(0) = A_1 (-\zeta + \sqrt{\zeta^2 - 1}) \omega_n + A_2 (-\zeta - \sqrt{\zeta^2 - 1}) \omega_n \]

or
\[ A_2 \omega_n \left[-\zeta + \sqrt{\zeta^2 - 1} + \zeta + \sqrt{\zeta^2 - 1}\right] = \]

\[ \dot{u}(0) + \left(\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n u(0) \]

or
\[ A_2 = \frac{\dot{u}(0) + \left(\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n u(0)}{2 \sqrt{\zeta^2 - 1} \omega_n} \]  
(f)

Substituting Eq. (f) in Eq. (d) gives
\[ A_2 = \frac{-\dot{u}(0) + \left(-\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n u(0)}{2 \sqrt{\zeta^2 - 1} \omega_n} \]

The solution, Eq. (c), now reads:
\[ u(t) = e^{-\zeta \omega_n t} \left(A_1 e^{-\omega_n t} + A_2 e^{\omega_n t}\right) \]

where
\[ \omega' = \sqrt{\omega^2 - 1} \omega_n \]

\[ A_1 = \frac{-\dot{u}(0) + \left(-\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n u(0)}{2 \omega'} \]

\[ A_2 = \frac{-\dot{u}(0) + \left(\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n u(0)}{2 \omega'} \]  
(g)
Problem 2.13

Given:

\( w = 20.03 \text{ kips (empty)} \), \( m = 0.0519 \text{ kip-sec}^2/\text{in.} \)

\( k = 2 \times (8.2) = 16.4 \text{ kips/in.} \)

\( c = 0.0359 \text{ kip-sec/in.} \)

(a) \( T_n = \frac{2\pi}{\sqrt{k}} \sqrt{\frac{m}{k}} = \frac{2\pi}{\sqrt{16.4}} \times \sqrt{\frac{0.0519}{16.4}} = 0.353 \text{ sec} \)

(b) \( \zeta = \frac{c}{2\sqrt{km}} = \frac{0.0359}{2\sqrt{(16.4)(0.0519)}} = 0.0194 \)

= 1.94\%
Problem 2.14

(a) The stiffness coefficient is

\[ \frac{3000}{2} = 1500 \text{ lb/in.} \]

The damping coefficient is

\[ c = 2\sqrt{km} \]

\[ c = 2\sqrt{\frac{1500 \cdot 3000}{386}} = 215.9 \text{ lb-sec/in.} \]

(b) With passengers the weight is w = 3640 lb. The damping ratio is

\[ \zeta = \frac{c}{2\sqrt{km}} = \frac{215.9}{2\sqrt{\frac{1500 \cdot 3640}{386}}} = 0.908 \]

(c) The natural vibration frequency for case (b) is

\[ \omega_D = \omega_n \sqrt{1 - \zeta^2} \]

\[ = \sqrt{\frac{1500}{3640/386}} \sqrt{1 - (0.908)^2} \]

\[ = 12.61 \times 0.419 \]

\[ = 5.28 \text{ rads/sec} \]
Problem 2.19

For motion of the building from left to right, the governing equation is
\[ m\ddot{u} + ku = -F \quad \text{(a)} \]
for which the solution is
\[ u(t) = A_2 \cos \omega_n t + B_2 \sin \omega_n t - u_F \quad \text{(b)} \]
With initial velocity of \( \dot{u}(0) \) and initial displacement \( u(0) = 0 \), the solution of Eq. (b) is
\[ u(t) = \frac{\dot{u}(0)}{\omega_n} \sin \omega_n t + u_F (\cos \omega_n t - 1) \quad \text{(c)} \]
\[ \dot{u}(t) = \dot{u}(0) \cos \omega_n t - u_F \omega_n \sin \omega_n t \quad \text{(d)} \]
At the extreme right, \( \dot{u}(t) = 0 \); hence from Eq. (d)
\[ \tan \omega_n t = \frac{\dot{u}(0)}{\omega_n} \frac{1}{u_F} \quad \text{(e)} \]
Substituting \( \omega_n = 4\pi \), \( u_F = 0.15 \text{ in.} \) and \( \dot{u}(0) = 20 \text{ in./sec} \) in Eq. (e) gives
\[ \tan \omega_n t = \frac{20}{4\pi} \frac{1}{0.15} = 10.61 \]
or
\[ \sin \omega_n t = 0.9956; \quad \cos \omega_n t = 0.0938 \]
Substituting in Eq. (c) gives the displacement to the right:
\[ u = \frac{20}{4\pi} (0.9956) + 0.15 (0.0938 - 1) = 1.449 \text{ in.} \]
After half a cycle of motion the amplitude decreases by
\[ 2u_F = 2 \times 0.15 = 0.3 \text{ in.} \]
Maximum displacement on the return swing is
\[ u = 1.449 - 0.3 = 1.149 \text{ in.} \]
Problem 3.4

(a) Machine running at 20 rpm.

\[ \frac{\omega}{\omega_n} = \frac{20}{200} = 0.1 \]

\[ u_o = \frac{(u_{\omega})_o}{\sqrt{1 - (\frac{\omega}{\omega_n})^2}} \]  

\[ or \]

\[ 0.2 = \frac{(u_{\omega})_o}{\sqrt{1 - (0.1)^2}} \Rightarrow (u_{\omega})_o = 0.1980 \text{ in.} \]

For \( \zeta = 0.25 \) and \( \omega/\omega_n = 0.1 \),

\[ u_o = (u_{\omega})_o \frac{1}{\sqrt{1 - (\frac{\omega}{\omega_n})^2 + [2 \zeta \omega/\omega_n]^2}} \]  

\[ or \]

\[ u_o = 0.1980 \frac{1}{\sqrt{1 - (0.1)^2 + [2 (0.25) (0.1)]^2}} \]

\[ = 0.1997 \text{ in.} \]

(b) Machine running at 180 rpm.

\[ \frac{\omega}{\omega_n} = \frac{180}{200} = 0.9 \]

From Eq. (a),

\[ 1.042 = \frac{(u_{\omega})_o}{\sqrt{1 - (0.9)^2}} \Rightarrow (u_{\omega})_o = 0.1980 \text{ in.} \]

For \( \zeta = 0.25 \) and \( \omega/\omega_n = 0.9 \), Eq. (b) reads

\[ u_o = 0.1980 \frac{1}{\sqrt{1 - (0.9)^2 + [2 (0.25) (0.9)]^2}} \]

\[ = 0.4053 \text{ in.} \]

(c) Machine running at 600 rpm.

\[ \frac{\omega}{\omega_n} = \frac{600}{200} = 3 \]

From Eq. (a),

\[ 0.0248 = \frac{(u_{\omega})_o}{\sqrt{1 - (3)^2}} \Rightarrow (u_{\omega})_o = 0.1980 \text{ in.} \]

For \( \zeta = 0.25 \) and \( \omega/\omega_n = 3 \), Eq. (b) reads

\[ u_o = 0.1980 \frac{1}{\sqrt{1 - (3)^2 + [2 (0.25) (3)]^2}} \]

\[ = 0.0243 \text{ in.} \]

(d) Summarizing these results together with given data:

<table>
<thead>
<tr>
<th>( \omega/\omega_n )</th>
<th>( (u_o)_{\zeta = 0} )</th>
<th>( (u_o)_{\zeta = 0.25} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>0.1997</td>
</tr>
<tr>
<td>0.9</td>
<td>1.042</td>
<td>0.4053</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0248</td>
<td>0.0243</td>
</tr>
</tbody>
</table>

The isolator is effective at \( \omega/\omega_n = 0.9 \); it reduces the deformation amplitude to 39% of the response without isolators. At \( \omega/\omega_n = 0.1 \) or 3, the isolator has essentially no influence on reducing the deformation.
Problem 3.5

Given:

\[ w = 1200 \text{ lbs, } \quad E = 30 \times 10^6 \text{ psi,} \]

\[ l = 10 \text{ in.}^4, \quad L = 8 \text{ ft; } \quad \zeta = 1\% \]

\[ p_0 = 60 \text{ lbs; } \quad \omega = \left(\frac{300}{60}\right)2\pi = 10\pi \text{ rad/s} \]

Stiffness of two beams:

\[ k = 2 \left(\frac{48EI}{L^4}\right) = 32,552 \text{ lbs/in.} \]

Natural frequency:

\[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{w/g}} = \sqrt{\frac{32,552}{1200/386}} = 102.3 \text{ rad/s} \]

Steady state response:

\[ R_d = \frac{1}{\sqrt[3]{1 - (\omega / \omega_n)^2 + [2\zeta \omega / \omega_n]^2}} \]

where \( \omega / \omega_n = 10\pi / 102.3 = 0.3071 \). Therefore,

\[ R_d = \frac{1}{\sqrt[3]{1 - 0.0943^2 + [2 \times 0.01 \times 0.3071]^2}} = 1.104 \]

Displacement:

\[ u_o = (u_d)_o R_d = \frac{p_o}{k} R_d \]

\[ = \frac{60}{32,552} \times 1.104 = 2.035 \times 10^{-3} \text{ in.} \]

Acceleration amplitude:

\[ \ddot{u}_o = \omega^2 u_o = (10\pi)^2 2.035 \times 10^{-3} \]

\[ = 2.009 \text{ in./sec}^2 = 0.0052 \text{ g} \]
Problem 3.6

In Eq. (3.2.1) replacing the applied force by \( p_0 \cos \omega t \) and dividing by \( m \) we get

\[
\ddot{u} + 2\zeta \omega_n \dot{u} + \omega_n^2 u = \frac{p_0}{m} \cos \omega t
\]  \hspace{1cm} (a)

(a) The particular solution is of the form:

\[ u_p(t) = C \sin \omega t + D \cos \omega t \] \hspace{1cm} (b)

Differentiating once and then twice gives

\[ \dot{u}_p(t) = C \omega \cos \omega t - D \omega \sin \omega t \] \hspace{1cm} (c)

\[ \ddot{u}_p(t) = -C \omega^2 \sin \omega t - D \omega^2 \cos \omega t \] \hspace{1cm} (d)

Substituting Eqs. (b)-(d) in Eq. (a) and collecting terms:

\[
\left[ \omega_n^2 - \omega^2 \right] C + 2\zeta \omega_n \omega D \sin \omega t \\
+ \left[ 2\zeta \omega_n \omega C + (\omega_n^2 - \omega^2) D \right] \cos \omega t = \frac{p_0}{m} \cos \omega t
\]

Equating coefficients of \( \sin \omega t \) and of \( \cos \omega t \) on the two sides of the equation:

\[
(\omega_n^2 - \omega^2) C - (2\zeta \omega_n \omega) D = 0 \] \hspace{1cm} (e)

\[ (2\zeta \omega_n \omega) C + (\omega_n^2 - \omega^2) D = \frac{p_0}{m} \] \hspace{1cm} (f)

Solving Eqs. (e) and (f) for \( C \) and \( D \) gives

\[ C = \frac{p_0}{m} \frac{2\zeta \omega_n \omega}{(\omega_n^2 - \omega^2) + (2\zeta \omega_n \omega)^2} \] \hspace{1cm} (g)

\[ D = \frac{p_0}{m} \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2) + (2\zeta \omega_n \omega)^2} \] \hspace{1cm} (h)

Substituting Eqs. (g) and (h) in Eq. (b) gives

\[ u_p(t) = \frac{p_0}{k} \left[ 1 - (\omega_n^2) \right] \cos \omega t + \frac{[2\zeta \omega_n \omega] \sin \omega t}{\left[ 1 - (\omega_n^2) \right]^2 + [2\zeta \omega_n \omega]^2} \]

(b) Maximum deformation is \( u_o = \sqrt{C^2 + D^2} \).

Substituting for \( C \) and \( D \) gives

\[ u_o = \frac{p_0}{k} \frac{1}{\sqrt{[1 - (\omega_n^2)]^2 + [2\zeta \omega_n \omega]^2}} \]

This is same as Eq. (3.2.11) for the amplitude of deformation due to sinusoidal force.