1. In class we showed that if the speed $U$, size $L$, and material properties are taken as controls in a reaction-diffusion-advection system, the governing equations can be written in dimensionless form, and depend only on three dimensionless parameters: $Re = UL/\nu$, $Pe = UL/D$, and $\gamma = U/L\nu$. Now consider thermal convection instead. The Boussinesq equations are

$$\frac{\partial u_j}{\partial x_j} = 0,$$

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x_j} + \frac{\rho'}{\rho_0} g_j + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{k}{c_p (\rho_0 + \rho')} \frac{\partial^2 T}{\partial x_i \partial x_i}.$$

Here $p' = p - p_0$ is the fluctuation from the reference pressure $p_0$, and $\rho' = \rho - \rho_0$ is the fluctuation from reference density $\rho_0$. If speed $U$, size $L$, temperature $T_0$, reference pressure $p_0$, and material properties (reference density $\rho_0$, viscosity $\nu$, and the ratio $k/C_p$ of thermal conductivity to specific heat) are taken as controls, how many dimensionless parameters characterize the system, according to the Buckingham Pi theorem? Write the Boussinesq equations in dimensionless form and identify the relevant dimensionless parameters. Stating their mathematical form is more important than knowing their historical names. (6 points)

2. The Burgers vortex in cylindrical coordinates has velocity components

$$u_r = -\gamma r,$$

$$u_\varphi = \frac{\Gamma}{2\pi r} \left(1 - e^{-\gamma r^2/2\nu}\right),$$

$$u_z = 2\gamma z,$$

where $\nu$ is the kinematic viscosity and $\gamma$ is a positive constant.

(a) What is the vorticity field of this flow? (3 points)
(b) Compute each term in the viscous vorticity equation and show that the equation is indeed satisfied. (4 points)
(c) Give a physical interpretation of this balance in terms of vortex stretching and diffusion and advection of vorticity. (3 points)

3. (a) For two-dimensional cylindrical coordinates with circular streamlines, show that the only nonzero component of vorticity, $\omega = \omega_z(r,t)$, satisfies the equation (4 points)

$$\frac{\partial \omega}{\partial t} = \nu \left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r}\right).$$

(b) Solve for $\omega_z(r,t)$ if the flow at $t = 0$ is a point vortex of circulation $\Gamma_0$ which is then allowed to decay. Show that the azimuthal velocity is given by (8 points)

$$u_\varphi = \frac{\Gamma_0}{2\pi r} \left(1 - e^{-\gamma r^2/\nu t}\right).$$

4. A circular disk of radius $a$ is parallel to and at a distance $h$ from a rigid plane, and the space between them is occupied by a viscous fluid. The pressure at the edge of the disk is atmospheric. Consider the case $h \ll a$, in which you may assume that spatial gradients in the direction normal to the disk are much larger than gradients parallel to the disk. Show that motion of the disk normal to the plane gives rise to a force on the disk in that direction with magnitude

$$-\frac{3\pi}{2} \frac{\mu a^4}{h^3} \frac{dh}{dt},$$

provided that $h \ll a$ and $\frac{\rho h^4}{\mu} \frac{|dh|}{dt} \ll 1$. 

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Hence, show that a constant force $F$ applied to the disk will pull it well away from the plane in a time

$$\frac{3 \pi \mu a^4}{4 \frac{\pi}{h^2} F}.$$  

(You may assume that the flow is steady and slow.) The fact that this time is large when $h$ is small is the basis of the phenomenon of viscous adhesion, used in adhesives like Scotch tape and the wringing together of accurately ground metal surfaces. (10 points)

5. An infinitely long solid circular cylinder of radius $a$ is immersed in a viscous fluid of infinite extent. The cylinder rotates at constant angular velocity $\Omega$ about its axis.

   (a) Find the resulting steady motion of the fluid, assuming that it is potential flow. (6 points)
   (b) Use your solution from part (a) to show that this motion is irrotational. (4 points)
   (c) Find the circulation about the cylinder. (4 points)

6. Watch the film [Low Reynolds Number Flow](https://example.com) made by the National Committee for Fluid Mechanics, and write a short essay (~500 words) summarizing its main points and connecting it to what we have covered in course lectures. (20 points)