Problem Set One Solutions

The first problem can certainly be worked by hand, but I'm doing it in Mathematica as an introduction to Mathematica. I'll try to explain what I am doing as I go along. I will do the second one by hand.

Step 1 is to write the motion of the particle before it hits the ground. I am assuming that \( w_0 \), the initial vertical velocity, is positive, but this is not necessary. (I'll leave you to think about that.)

\[
\begin{align*}
y &= v_0 t \\
z &= z_0 + w_0 t - \frac{1}{2} g t^2 \\
t v_0 \\
&= -\frac{g t^2}{2} + t w_0 + z_0
\end{align*}
\]

Differentiate to get the particle speed.

\[
\begin{align*}
v &= D[y, t] \\
w &= D[z, t] \\
v_0 \\
&= -g t + w_0
\end{align*}
\]

The maximum height of the particle occurs when \( w = 0 \), and so I have by inspection

\[
\begin{align*}
t_M &= \frac{w_0}{g} \\
&= \frac{w_0}{g}
\end{align*}
\]

I can start the problem over at this point. This is not necessary, but it helps if you are working the problem by hand. tprime is now the time variable, counting from the time of maximum height.

\[
\begin{align*}
t &= t_M + tprime \\
tprime &= \frac{w_0}{g}
\end{align*}
\]

Find the value of tprime when the mass hits the ground, and clean it up. The Simplify command assumptions that all the variables are complex and can give very unhelpful expressions. If I tell it that everything is positive, it will give me a nice simple expression that I can use.

\[
\begin{align*}
\text{Solve}[z == 0, tprime] \\
\{\{tprime \to -\frac{i \sqrt{-w_0^2 - 2 g z_0}}{\sqrt{g}}\}, \{tprime \to \frac{i \sqrt{-w_0^2 - 2 g z_0}}{\sqrt{g}}\}\}
\end{align*}
\]
Simplify[%, Assumptions \[\rightarrow\] \( g > 0 \&\& w0 > 0 \&\& z0 > 0 \)]
\[
\{\{tprime \rightarrow \frac{\sqrt{w0^2 + 2 g z0}}{g}\}, \{tprime \rightarrow -\frac{\sqrt{w0^2 + 2 g z0}}{g}\}\}
\]

The first root is positive and the second negative. I want the positive root. The % means the previous item, [[1]] means the first element, and /. means to substitute. This gives me tp1, the value of tprime when the mass strikes the ground, without changing tprime at all.

\[
\sqrt{w0^2 + 2 g z0}
\]
\[
\frac{g}{\sqrt{w0^2 + 2 g z0}}
\]

I simplify y, z, v and w for convenience.

\[
y = \text{Simplify}[y]
\]
\[
z = \text{Simplify}[z]
\]
\[
v0 \left(tprime + \frac{w0}{g}\right) - \frac{g tprime^2}{2} + \frac{w0^2}{2 g} + z0
\]

\[
v = \text{Simplify}[v]
\]
\[
w = \text{Simplify}[w]
\]
\[
v0 - g tprime
\]

The momentum just before the mass strikes the ground is given by

\[
pminus = m \{0, v, w\} /. tprime \rightarrow tp1
\]
\[
\{0, m v0, -m \sqrt{w0^2 + 2 g z0}\}
\]

Changing the sign of the z component gives the momentum just after striking the ground

\[
pplus = pminus / . pminus[[3]] \rightarrow -pminus[[3]]
\]
\[
\{0, m v0, m \sqrt{w0^2 + 2 g z0}\}
\]

The impulse is the difference between these.

\[
pimpulse = pplus - pminus
\]
\[
\{0, 0, 2 m \sqrt{w0^2 + 2 g z0}\}
\]

The impulse is the same at every bounce because I have supposed that energy is conserved. Thus the change in momentum at the second bounce will be the same as the first, and the change in angular momentum will be \( r \times \Delta p \) and the radius vector is just y. The second strike takes place at tprime = 3 tp1, so
\[ y_2 = y / t_{\text{prime}} \rightarrow 3 t_{\text{pl}} \]

\[ v_0 \left( \frac{w_0}{g} + \frac{3 \sqrt{w_0^2 + 2 gz_0}}{g} \right) \]

\[ \text{Limpulse} = y_2 \text{pimpulse} \]

\[ \{ 0, 0, 2 m v_0 \sqrt{w_0^2 + 2 gz_0} \left( \frac{w_0}{g} + \frac{3 \sqrt{w_0^2 + 2 gz_0}}{g} \right) \} \]

and you can look at various arrangements of the terms.

\[ \text{FullSimplify}[\%] \]

\[ \{ 0, 0, \frac{2 m v_0 \sqrt{w_0^2 + 2 gz_0}}{g} \left( w_0 + 3 \sqrt{w_0^2 + 2 gz_0} \right) \} \]

\[ \text{Expand}[\%] \]

\[ \{ 0, 0, \frac{6 m v_0 w_0^2}{g} + 12 m v_0 z_0 + \frac{2 m v_0 w_0 \sqrt{w_0^2 + 2 gz_0}}{g} \} \]

\[ \text{Collect}[\%, g, \text{Simplify}] \]

\[ \{ 0, 0, 12 m v_0 z_0 + \frac{2 m v_0 w_0 \left( 3 w_0 + \sqrt{w_0^2 + 2 gz_0} \right)}{g} \} \]
2. Denote the Earth-Sun distance by \( D \) and the Earth-Moon distance by \( d \). The motion of the Earth can be written

\[
\begin{bmatrix}
D \\
d \\
0
\end{bmatrix}
= \mathbf{T}
\begin{bmatrix}
\cos \omega_t t \\
\sin \omega_t t \\
0
\end{bmatrix}
\]

where \( \omega = 2\pi/365 \text{ day}^{-1} \).

The motion of the Moon can be written

\[
\begin{bmatrix}
D \\
d \\
0
\end{bmatrix}
= \mathbf{D}
\begin{bmatrix}
\cos (\omega t + \phi) \\
\sin (\omega t + \phi) \\
0
\end{bmatrix}
+ \begin{bmatrix}
D \\
d \\
0
\end{bmatrix}
\]

where \( \omega = 2\pi/24 \text{ day}^{-1} \).

(The Moon's orbit is actually inclined a bit. I didn't expect you to add that in, but it is wonderful if you did.)

The momenta are

\[
P_e = M_{\text{E}} \dot{r}_e = M_{\text{E}} \mathbf{D}
\begin{bmatrix}
-\sin \omega_t t \\
\cos \omega_t t \\
0
\end{bmatrix}
\]

\[
P_m = M_{\text{M}} \dot{r}_m = m_{\text{M}} \mathbf{D}
\begin{bmatrix}
-\sin (\omega t + \phi) \\
\cos (\omega t + \phi) \\
0
\end{bmatrix}
\]

\[
+ M_{\text{E}} \mathbf{D}
\begin{bmatrix}
-\sin \omega t \\
\cos \omega t \\
0
\end{bmatrix}
\]
The angular moments are then

\[ L_E = r_E \times p_E = \begin{vmatrix} i & j & k \\ \cos \phi t & \sin \phi t & 0 \\ -\sin \phi t & \cos \phi t & 0 \end{vmatrix} \]

\[ L_E = M \Omega^2 \phi \]

\[ L_m = r_m \times p_E = \begin{vmatrix} i & j & k \\ d\cos (\omega t + \phi) + D \cos \omega t \\ d\sin (\omega t + \phi) + D \sin \omega t \end{vmatrix} \]

\[ = \begin{vmatrix} d\cos (\omega t + \phi) + D \cos \omega t \\ d\sin (\omega t + \phi) + D \sin \omega t \end{vmatrix} \]

\[ = m \omega d^2 + M \Omega^2 \phi^2 + 2 m \omega d D (\cos \omega t \cos (\omega t + \phi) + \sin \omega t \sin (\omega t + \phi)) \]

\[ L_m = m \omega d^2 + M \Omega^2 \phi^2 + 2 m \omega d D \cos \phi (\Omega - \omega) t - d \phi \]

and the total angular momentum is their sum.