Lecture 16: Control of the Three Link Robot
Start from square one using what we know so far

all the holonomic constraints

generalized coordinates

Lagrangian

generalized forces

goal

Euler-Lagrange track   Hamilton track

linearization \hspace{1cm} \text{equilibrium forces}

control design   simulation (nonlinear)
We’ll see this as we go along in the Mathematica notebook, but

surviving generalized coordinates: \( q = \begin{pmatrix} \psi_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \)

the generalized forces: \( Q = \{ \tau_{01} \quad \tau_{12} - \tau_{23} \quad \tau_{23} \} \)

Note that the forces (torques) are such as to increase the corresponding angles

the goal: \( q = \begin{pmatrix} \psi_{1d} \\ \theta_{2d} \\ \theta_{3d} \end{pmatrix} \), \( u = \begin{pmatrix} \dot{\psi}_{1d} \\ \dot{\theta}_{2d} \\ \dot{\theta}_{3d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \)
We’ll look at the Hamiltonian track all the way through

We’ll use the Euler-Lagrange track to find a linear control

We’ll put the linear control into the Hamiltonian simulation to assess it
We are going to have to make some assumptions about the gains

There can be many more gains than there are exponents

I’ll address this in the context of the actual problem
In this case, and in many cases where there are as many inputs as degrees of freedom, an alternative analysis leads to a nonlinear control, good for all initial conditions.

**feedback linearization**

The key is the ability to solve equations for the Euler-Lagrange accelerations for the forces.

I’ll talk a little in general, and then we’ll see how this can be fit into our analytical pattern.
Suppose there are as many external forces as there are degrees of freedom

(Robots are like this.)

Then we can often devise a control for the nonlinear system without approximation

I don’t want to go too far down this road, but it fits so nicely into what we have been doing that I’d like to mention it. It’s not in the book.
Look at a one degree of freedom system — the Euler-Lagrange equation is

\[ \ddot{q} = f(q, \dot{q}) + b(q) f_E \]

\( f \) is nonlinear in its arguments, as may be \( b \). I require that \( b \) be nonsingular.

\[ \ddot{q} = f(q, \dot{q}) + b(q) f_E = -2\zeta \omega_n \dot{q} - \omega_n^2 (q - q_d) \]

\[ f_E = -\frac{f(q, \dot{q}) + 2\zeta \omega_n \dot{q} + \omega_n^2 (q - q_d)}{b(q)} \]

If I can measure \( q \) and its derivative and \( b \) is nonsingular, then I have a universal control that will drive \( q \) to \( q_d \).
Here’s what we have done, and its consequence

\[ \ddot{q} = f(q, \dot{q}) + b(q) f_E = -2\zeta \omega_n \dot{q} - \omega_n^2 (q - q_d) \]

\[ f_E = -\frac{f(q, \dot{q}) + 2\zeta \omega_n \dot{q} + \omega_n^2 (q - q_d)}{b(q)} \]

\[ \ddot{q} + 2\zeta \omega_n \dot{q} + \omega_n^2 (q - q_d) = 0 \]

which has the usual decay (perhaps oscillatory) to \( q_d \)

**How do we do this in general?**
We can take an Euler-Lagrange approach to this

\[ M_{ij} \ddot{q}^j + \frac{1}{2} \frac{d}{dt} \left( M_{ij} \dot{q}^j \right) = \frac{\partial L}{\partial q^i} + Q_i \]

\[ Q_i = \frac{\partial \dot{W}}{\partial \dot{q}^i} = a_{ij}(q^k) f_E^j \]

\[ M_{ij} \ddot{q}^j + \frac{1}{2} \frac{d}{dt} \left( M_{ij} \dot{q}^j \right) = \frac{\partial L}{\partial q^i} + a_{ij}(q^k) f_E^j \]

\[ \ddot{q}^m = -\frac{1}{2} \overline{M}^{mi} \frac{d}{dt} \left( M_{ij} \dot{q}^j \right) + \overline{M}^{mi} \frac{\partial^2 L}{\partial q^i} + \overline{M}^{mi} a_{ij}(q^k) f_E^j \]
\[
\ddot{q}^m = -\frac{1}{2} \overline{M}^{mi} d\left( M_{ij} \right) \dot{q}^j + \overline{M}^{mi} \frac{\partial^2 L}{\partial q^i} + \overline{M}^{mi} a_{ij}(q^k) f_{E}^j
\]

\[
\ddot{q}^m = -\frac{1}{2} \overline{M}^{mi} \frac{d}{dt}(M_{ij}) \dot{q}^j + \overline{M}^{mi} \frac{\partial^2 L}{\partial q^i} + \overline{M}^{mi} a_{ij}(q^k) f_{E}^j = -2\varsigma \omega_n \dot{q}^m - \omega_n^2 q^m
\]

these can be different for each coordinate

\[
\overline{M}^{mi} a_{ij}(q^k) f_{E}^j = \frac{1}{2} \overline{M}^{mi} \frac{d}{dt}(M_{ij}) \dot{q}^j - \overline{M}^{mi} \frac{\partial^2 L}{\partial q^i} - 2\varsigma \omega_n \dot{q}^m - \omega_n^2 q^m
\]

\[
a_{ij}(q^k) f_{E}^j = \frac{1}{2} \frac{d}{dt}(M_{ij}) \dot{q}^j - \frac{\partial^2 L}{\partial q^i} - M_{im} \left( 2\varsigma \omega_n \dot{q}^m + \omega_n^2 q^m \right)
\]

which I can solve (usually) for the components of \( f_E \).
These forces/torques can then be put into the simulation as before

It’s probably worth noting that this approach allows us to track time-dependent goals!