Lecture 12: The Constraint Null Space

Nonholonomic (including the pseudo ones) constraints

\[ \mathbf{Cq} = 0 \iff C^i_j \dot{q}^j = 0 \]

The matrix \( \mathbf{C} \) has \( M \) rows and \( N \) columns, \( M < N \)

If we have set this up correctly, the rows of \( \mathbf{C} \) are independent

(If not, do a row reduction and rewrite the constraints)
Each row is an $N$ row vector

The rate of change of $\mathbf{q}$ must be perpendicular to all the row vectors

There can be $K = N - M$ independent vectors perpendicular to the row vectors of $\mathbf{C}$

These vectors form the **null space** of $\mathbf{C}$

Let’s try to say this another way
\( \dot{q} \) lives in an \( N \) dimensional vector space

The rows of \( C \) live in the same space

The \( M \) rows of \( C \) form a partial basis in that space

\( \dot{q} \) must be perpendicular to those basis vectors

The vectors that are perpendicular to the rows of \( C \) make up the null space
there will be \( K \) of them, call them \( s_1, s_2, \) etc.

\[
\dot{q} = u_1 s_1 + u_2 s_2 + \cdots \iff \dot{q}^j = S_k^j u^k
\]
\( \mathbf{u} \) is a new vector; it is **not** the \( \mathbf{u} \) vector we used in the Euler-Lagrange method.

\[ \mathbf{S} \] is an \( N \times K \) matrix. Its columns are the basis of the null space.

Let’s do a simple example of this before seeing how to use it.
Let’s look at the erect wheel

Simple holonomic constraints \( z = r_w, \quad \theta = -\frac{\pi}{2} \)

Generalized coordinates \( \mathbf{q} = \begin{bmatrix} x \\ y \\ \phi \\ \psi \end{bmatrix} \)

Rolling constraints \( \dot{x} = r_w \dot{\psi} \cos \phi, \quad \dot{y} = r_w \dot{\psi} \sin \phi \)

\( \dot{q}^1 - r_w \dot{q}^4 \cos q^3 = 0 = \dot{q}^2 - r_w \dot{q}^4 \sin q^3 \)
The constraint matrix

\[
C = \begin{bmatrix}
1 & 0 & 0 & -r_w \cos \phi \\
0 & 1 & 0 & -r_w \sin \phi \\
\end{bmatrix}
\]

It’s obviously of full rank (rank = 2)

There are two vectors in the null space
we can work them out on the blackboard
and combine them in the \( S \) matrix

\[
S = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]
The rate of change of $\mathbf{q}$ becomes

$$\dot{\mathbf{q}} = S\mathbf{u} = \begin{bmatrix} r_w \cos \phi & 0 \\ r_w \sin \phi & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

If you expand this you will see that $u^1$ and $u^2$ denote the two rotation rates.

The dimension of $\mathbf{u}$ is the true number of degrees of freedom taking the nonholonomic constraints into account.
Let’s move on and see where this can take us.

the momentum equations

\[ \dot{p}_i = \frac{\partial L}{\partial \dot{q}^i} + \lambda_j C^j_i + Q_i \]

the momentum

\[ p_i = M_{ij} \dot{q}^j = M_{ij} S^j_k \dot{u}^k \]

\( M_{ij} S^j_k \) is a function only of the \( q \)s

combine in two steps

\[ \dot{p}_i = M_{ij} S^j_k \dot{u}^k + \frac{d}{dt} \left( M_{ij} S^j_k \right) \dot{u}^k = M_{ij} S^j_k \dot{u}^k + \frac{\partial}{\partial q^m} \left( M_{ij} S^j_k \right) q^m \dot{u}^k \]

\[ \dot{p}_i = M_{ij} S^j_k \dot{u}^k + \frac{\partial}{\partial q^m} \left( M_{ij} S^j_k \right) S^m_n \dot{u}^n u^k \]
combine the momentum equation

$$\dot{p}_i = \frac{\partial L}{\partial q^i} + \lambda_i C^j_i + Q_i$$

with what we just did

$$\dot{p}_i = M_{ij} S^j_k \dot{u}^k + \frac{\partial (M_{ij} S^j_k)}{\partial q^m} S^m_n u^a u^k$$

momentum equation
in terms of $u$

$$M_{ij} S^j_k \dot{u}^k = -\frac{\partial (M_{ij} S^j_k)}{\partial q^m} S^m_n u^a u^k + \frac{\partial L}{\partial q^i} + \lambda_j C^j_i + Q_i$$

And finally we can get rid of the Lagrange multipliers

$$M_{ij} S^j_k \dot{u}^k S^i_p = -\frac{\partial (M_{ij} S^j_k)}{\partial q^m} S^m_n u^a u^k S^i_p + \frac{\partial L}{\partial q^i} S^i_p + \lambda_j C^j_i S^i_p + Q_i S^i_p$$

0!
Before multiplying by $S$ the free index was $i$, which runs from 1 to $N$

After multiplying by $S$, the free index becomes $p$ which runs from 1 to $K$

We have reduced the number of momentum equations to the number of degrees of freedom and gotten rid of the Lagrange multipliers.

$$M_{ij} S_k^j \dot{u}^k S_p^i = -\frac{\partial (M_{ij} S_k^j)}{\partial q^m} S_n^m u^n u^k S_p^i + \frac{\partial L}{\partial q^i} S_p^i + Q_i S_p^i$$

As usual, it is not as ghastly in practice as it looks in its full generality.
We can go back to the erect coin to see how some of this plays out

\[ L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} A \dot{\phi}^2 + \frac{1}{2} C \dot{\psi}^2 = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} A \dot{q}_3^2 + \frac{1}{2} C \dot{q}_4^2 \]

The momentum

\[ \dot{p}_1 = m \dot{q}_1, \quad p_2 = m \dot{q}_2, \quad p_3 = A \dot{q}_3, \quad p_4 = C \dot{q}_4 \]

The velocity equations

\[
\begin{bmatrix}
    r_w \cos \phi & 0 \\
    r_w \sin \phi & 0 \\
    0 & 1 \\
    1 & 0
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
    \dot{q}_1 = r_w \cos \phi u_1 \\
    \dot{q}_2 = r_w \sin \phi u_1 \\
    \dot{q}_3 = u_2 \\
    \dot{q}_4 = u_1
\end{bmatrix}
The momentum becomes

\[ p_1 = mr_w \cos \phi u_1, \quad p_2 = mr_w \sin \phi u_1, \quad p_3 = Au_2, \quad p_4 = Cu_1 \]

differentiate

\[ \dot{p}_1 = mr_w \cos q^3 u_1 - mr_w \sin q^3 u_1 u^2 \]
\[ \dot{p}_2 = mr_w \sin q^3 u_1 + mr_w \cos q^3 u_1 u^2 \]
\[ \dot{p}_3 = A\dot{u}_2, \quad \dot{p}_4 = C\dot{u}_1 \]

None of the coordinates appears in the Lagrangian, so the \(dL/dq\) term is zero and there are no generalized forces and multiplication by \(S\) removes the Lagrange multipliers

My equations are simply \(\dot{p}_i S^i_j = 0\)
\[ \dot{p}_i S^i_j = 0 \quad \text{Let’s build it} \]

\[
S = \begin{bmatrix}
    r_w \cos q^3 & 0 \\
    r_w \sin q^3 & 0 \\
    0 & 1 \\
    1 & 0
\end{bmatrix}
\]

\[
\dot{p}_i S^i_j = \begin{bmatrix}
    \dot{p}_1 \\
    \dot{p}_2 \\
    \dot{p}_3 \\
    \dot{p}_4
\end{bmatrix} = \begin{bmatrix}
    r_w \cos q^3 & 0 \\
    r_w \sin q^3 & 0 \\
    0 & 1 \\
    1 & 0
\end{bmatrix}
\]

expand

\[
r_w \cos q^3 \dot{p}_1 + r_w \sin q^3 \dot{p}_2 + \dot{p}_4 = 0
\]

\[
\dot{p}_3 = 0
\]

(continued on the next page)
\[ r_w \cos q^3 \dot{p}_1 + r_w \sin q^3 \dot{p}_2 + \dot{p}_4 = 0 \]
\[ \dot{p}_3 = 0 \]

substitute

\[ \dot{p}_1 = mr_w \cos q^3 u^1 - mr_w \sin q^3 u^1 u^2 \]
\[ \dot{p}_2 = mr_w \sin q^3 u^1 + mr_w \cos q^3 u^1 u^2 \]
\[ \dot{p}_3 = A \dot{u}^2, \quad \dot{p}_4 = C \dot{u}^1 \]

A lot of trigonometric cancellation leads to the final equations

\[ (C + mr_w^2) \dot{u}^1 = 0 \]
\[ A \dot{u}^2 = 0 \]
So, what is the drill?

Lagrangian

Holonomic constraints

Generalized coordinates

Nonholonomic constraints

Constraint matrix — null space matrix

\[ \dot{q}^j = S_k^j u^k \]

The velocity equations
Hamilton’s momentum equations

Eliminate \( p \) in favor of \( u \)

Multiply by \( S \) to reduce the number of momentum equations

Convert to explicit time dependence for the simulation
What’s new and what’s hard?

The idea of the null space is new and finding it can be hard

Mathematica can give you a null space, but it may not be what you want
I generally work out my own

The goal is to devise a null space such that
the components of $\mathbf{u}$ have a useful interpretation
Multiply \( \mathbf{C} \) and an arbitrary vector of length \( N \)

This will give you \( K \) conditions from which you can choose \( K \) basis vectors

Had I done this with the case just reviewed

\[
\mathbf{Cs} = \begin{bmatrix} 1 & 0 & 0 & -r_w \cos \phi \\ 0 & 1 & 0 & -r_w \sin \phi \end{bmatrix} \begin{bmatrix} s^1 \\ s^2 \\ s^3 \\ s^4 \end{bmatrix} = \begin{bmatrix} s^1 - r_w \cos \phi s^4 \\ s^2 - r_w \sin \phi s^4 \end{bmatrix} = 0
\]

\( s^1 \) is associated with \( x \), \( s^2 \) with \( y \), \( s^3 \) with \( \phi \) and \( s^4 \) with \( \psi \)

I’d like to have one vector for which \( s^3 \) is unity
and one for which \( s^4 \) is unity
And that’s what I did

\[
S = \begin{pmatrix}
  r_w \cos q^3 & 0 \\
  r_w \sin q^3 & 0 \\
  0 & 1 \\
  1 & 0
\end{pmatrix}
\]

and that’s why the equations came out so nicely

(As it happens, this is the null space that Mathematica gives.)
Let’s look at the 3 x 6 matrix from the two wheel system

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{2}r_W \cos q^3 & -\frac{1}{2}r_W \cos q^3 \\
0 & 1 & 0 & 0 & -\frac{1}{2}r_W \sin q^3 & -\frac{1}{2}r_W \sin q^3 \\
0 & 0 & 1 & 0 & \frac{r_W}{l_R} & -\frac{r_W}{l_R}
\end{bmatrix}
\]

We can multiply this one out
Do the multiplication and look at the three components of the product

\[
Cs = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{2} r_w \cos q^3 & -\frac{1}{2} r_w \cos q^3 \\
0 & 1 & 0 & 0 & -\frac{1}{2} r_w \sin q^3 & -\frac{1}{2} r_w \sin q^3 \\
0 & 0 & 1 & 0 & \frac{r_w}{l_R} & -\frac{r_w}{l_R}
\end{bmatrix}
\begin{bmatrix}
s^1 \\ s^2 \\ s^3 \\ s^4 \\ s^5 \\ s^6
\end{bmatrix}
\]

\[
s^1 - \frac{1}{2} r_w \cos q^3 s^5 - \frac{1}{2} r_w \cos q^3 s^6 = 0
\]
\[
s^2 - \frac{1}{2} r_w \sin q^3 s^5 - \frac{1}{2} r_w \sin q^3 s^6 = 0
\]
\[
s^3 + \frac{r_w}{l_R} s^5 - \frac{r_w}{l_R} s^6 = 0
\]

We can pick any three and find the others
We can take the three spins as fundamental and select bases such that

\[ s^4 = 1, s^5 = 0, s^6 = 0; \]
\[ s^4 = 0, s^5 = 1, s^6 = 0; \]
\[ s^4 = 0, s^5 = 0, s^6 = 1 \]
\( s^4 = 1, \ s^5 = 0, \ s^6 = 0 \)

\[
\begin{align*}
    s^1 - \frac{1}{2} r_w \cos q^3 s^3 - \frac{1}{2} r_w \cos q^3 s^5 &= 0 \\
    s^2 - \frac{1}{2} r_w \sin q^3 s^5 - \frac{1}{2} r_w \sin q^3 s^6 &= 0 \\
    s^3 + \frac{r_w}{l_B} s^3 - \frac{r_w}{l_B} s^6 &= 0
\end{align*}
\]

\[\begin{pmatrix}
    s^1 \\
    s^2 \\
    s^3 \\
    s^4 \\
    s^5 \\
    s^6
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    1 \\
    0 \\
    0
\end{pmatrix}\]

and the base vector is
\[ s^4 = 0, \ s^5 = 1, \ s^6 = 0 \]

\[ s^1 - \frac{1}{2} r_w \cos q^3 s^5 - \frac{1}{2} r_w \cos q^3 s^6 = 0 \]
\[ s^2 - \frac{1}{2} r_w \sin q^3 s^5 - \frac{1}{2} r_w \sin q^3 s^6 = 0 \]
\[ s^3 + \frac{r_w}{l_R} s^5 - \frac{r_w}{l_R} s^6 = 0 \]

and the base vector is

\[
\begin{pmatrix}
    s^1 \\
    s^2 \\
    s^3 \\
    s^4 \\
    s^5 \\
    s^6
\end{pmatrix} =
\begin{pmatrix}
    1 \\
    0 \\
    0 \\
    \frac{1}{2} r_w \cos q^3 \\
    \frac{1}{2} r_w \sin q^3 \\
    -\frac{r_w}{l_R}
\end{pmatrix}
\]
\[ s^4 = 0, \ s^5 = 0, \ s^6 = 1 \]

\[ s^1 - \frac{1}{2} r_W \cos q^3 s^5 - \frac{1}{2} r_W \cos q^3 s^6 = 0 \]
\[ s^2 - \frac{1}{2} r_W \sin q^3 s^5 - \frac{1}{2} r_W \sin q^3 s^6 = 0 \]
\[ s^3 + \frac{r_W}{l_R} s^5 - \frac{r_W}{l_R} s^6 = 0 \]

and the base vector is

\[
\begin{bmatrix}
  s^1 \\
  s^2 \\
  s^3 \\
  s^4 \\
  s^5 \\
  s^6
\end{bmatrix} =
\begin{bmatrix}
  \frac{1}{2} r_W \cos q^3 \\
  \frac{1}{2} r_W \cos q^3 \\
  \frac{1}{2} r_W \sin q^3 \\
  r_W/l_R \\
  r_W/l_R \\
  1
\end{bmatrix}
\]
put it all together

$$S = \begin{bmatrix}
0 & \frac{1}{2}r_w \cos q^3 & \frac{1}{2}r_w \cos q^3 \\
0 & \frac{1}{2}r_w \sin q^3 & \frac{1}{2}r_w \sin q^3 \\
0 & -\frac{r_w}{l_R} & \frac{r_w}{l_R} \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

And the rank of this matrix is obviously three
The advantage of this is that there are fewer equations to deal with

The disadvantage is that you do not get the constraint forces (through the Lagrange multipliers)

The main consideration is then whether you need the constraint forces

If you believe that, whatever they are, the ground will be able to provide them then you may as well adopt the constraint null space method.

Let’s look at the two wheel cart using this method