

# ME 406

## Assignment #9 Solutions

`sysid`

Mathematica 6.0.3, DynPac 11.02, 4/8/2009

`intreset; plotreset; imsize = 250;`

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### ■ Problem 1

The equations are

$$\dot{x} = -x, \quad \dot{y} = x^2 + y, \quad \dot{z} = \frac{2}{3}x^2 + z.$$

Let  $F = x^2 + y + 3z$ . To show that the surface  $F = 0$  is an invariant surface, we have show that points on the surface remain on the surface -- that is,  $dF/dt$  evaluated on  $F = 0$  is 0. We have

$$\frac{dF}{dt} = \dot{x} F_x + \dot{y} F_y + \dot{z} F_z = -x(2x) + (x^2 + y)(1) + \left(\frac{2}{3}x^2 + z\right)(3) = x^2 + y + 3z,$$

which is zero when evaluated on  $F = 0$ .

---

### ■ Problem 2

We define the system for DynPac.

```
setstate[{x, y}]; setparm[{}]; slopevec = {-x + y^2, x + y + x y};
```

```
sysname = "Prob2NL";
```

```
eigval[{0, 0}]
```

```
{-1, 1}
```

The eigenvalues are -1 and 1, hence neither has a zero real part, so the equilibrium is hyperbolic. It is in fact a saddle point, as DynPac tells us also.

```
classify2D[{0, 0}]
```

Abbreviations used in classify2D.

L = linear, NL = nonlinear, R2 = repeated root.

Z1 = one zero root, Z2 = two zero roots.

This message printed once.

```
unstable - saddle
```

We now construct the linearized system. The matrix A of the linearized system is the derivative matrix evaluated at the equilibrium:

```
A = Dmatval[{0, 0}]
```

```
{{-1, 0}, {1, 1}}
```

We save our nonlinear system and then define the linear system for DynPac.

```
savesys[Prob2NL];
```

```
slopevec = A.statevec
```

```
{-x, x + y}
```

```
sysname = "Prob2LIN";
```

The manifolds of the linear system will be determined by the eigenvectors and eigenvalues.

```
Eigensystem[A]
```

```
{{-1, 1}, {{-2, 1}, {0, 1}}}
```

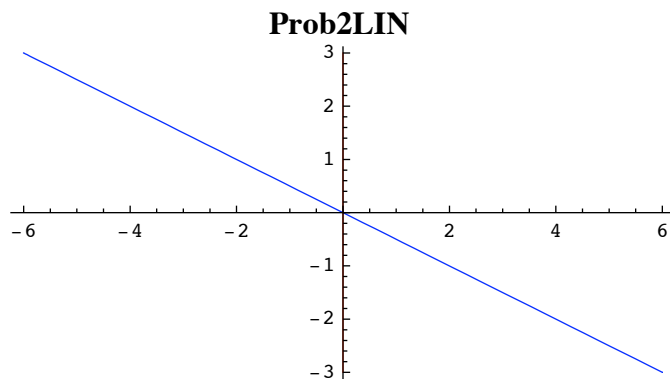
Thus the stable manifold (associated with the eigenvalue -1) is spanned by the vector  $\{-2, 1\}$ . The unstable manifold (associated with the eigenvalue 1) is spanned by  $\{0, 1\}$ . We plot these, using blue for stable, red for unstable.

```
man1 =
```

```
  ParametricPlot[{2 u, -u}, {u, -3, 3}, PlotStyle → RGBColor[0, 0, 1]];
```

```
man2 = ParametricPlot[{0, u}, {u, -3, 3}, PlotStyle → RGBColor[1, 0, 0]];
```

```
show[{man1, man2}]
```



It is actually just as easy to get these from DynPac by a numerical integration. We do that next.

```
plrange = {{-3.1, 3.1}, {-3.1, 3.1}}; asprat = 1; axon = False;
frameon = True; arrowflag = True; arrowvec = {1 / 2};
rangeflag = True; ranger = plrange;

t0 = 0.0; h = 0.02; nsteps = 800;

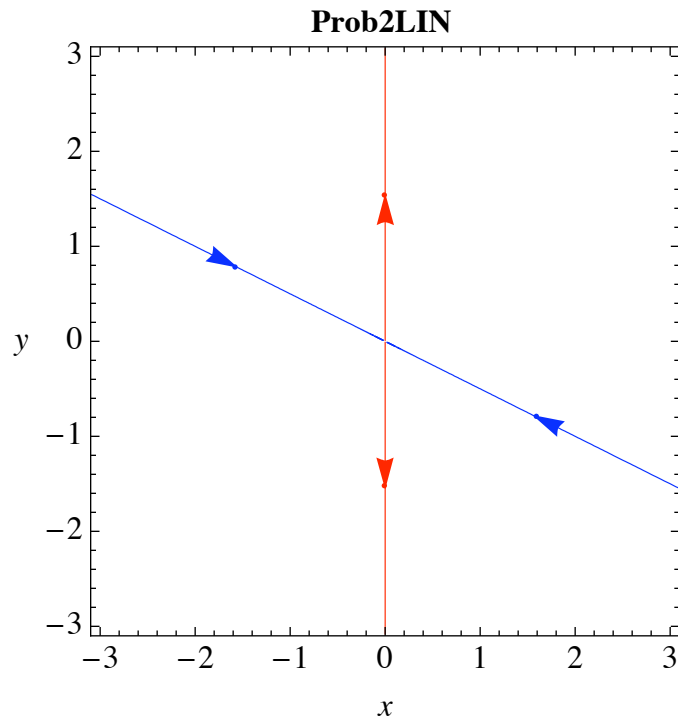
eq = {0, 0}; eigstab = {-2, 1}; eigunstab = {0, 1};

eps = 0.01; initset = {eq + eps * eigstab,
  eq - eps * eigstab, eq + eps * eigunstab, eq - eps * eigunstab}
{{-0.02, 0.01}, {0.02, -0.01}, {0, 0.01}, {0, -0.01}}

h = {-0.01, -0.01, 0.01, 0.01};

setcolor[{Blue, Blue, Red, Red}];
```

```
linport = portrait[initset, t0, h, nsteps, 1, 2]
```

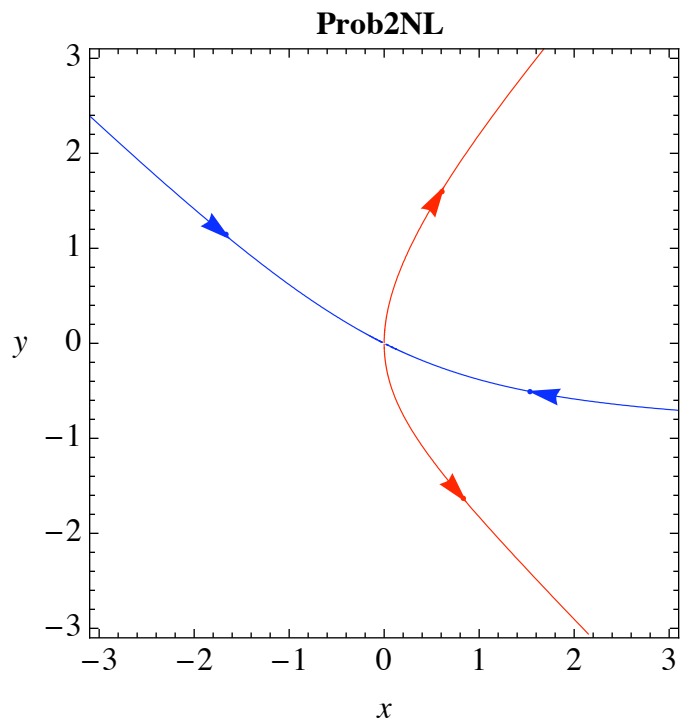


Now we carry out the same construction for the full nonlinear system.

```
restoresys[Prob2NL];
```

```
h = {-0.01, -0.01, 0.01, 0.01};
```

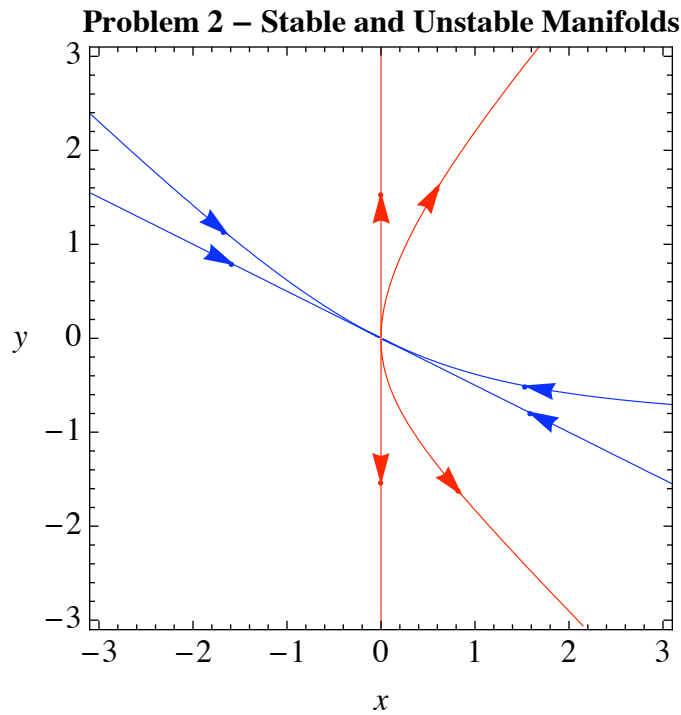
```
nlport = portrait[initset, t0, h, nsteps, 1, 2]
```



We combine these.

```
labon = "Problem 2 - Stable and Unstable Manifolds";
```

```
show[linport, nlport]
```



### ■ Problem 3

We enter the matrix A.

```
A = {{0, 1/4, 3/2, 3/4, -7/4},
      {3/2, 3/4, -1/2, -5/4, 11/4}, {1, -1/4, -1/2, -3/4, 7/4},
      {5, 19/4, -1/2, -15/4, 19/4}, {5/2, 5/2, 1, -2, 5/2}};
```

```
MatrixForm[A]
```

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{2} & \frac{3}{4} & -\frac{7}{4} \\ \frac{3}{2} & \frac{3}{4} & -\frac{1}{2} & -\frac{5}{4} & \frac{11}{4} \\ 1 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} & \frac{7}{4} \\ 5 & \frac{19}{4} & -\frac{1}{2} & -\frac{15}{4} & \frac{19}{4} \\ \frac{5}{2} & \frac{5}{2} & 1 & -2 & \frac{5}{2} \end{pmatrix}$$

We get the eigenvalues and eigenvectors of A.

**Eigensystem[A]**

$$\left\{ \{-2, -2, 1, 1, 1\}, \left\{ \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 1 \right\}, \{0, 0, 0, 0, 0\}, \{0, 1, 0, 2, 1\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\} \right\}$$

We see that there is an eigenvalue -2 of multiplicity 2 with one eigenvector, and an eigenvalue 1 of multiplicity 3, also with one eigenvector. We will have to use generalized eigenvectors. We start with the eigenvalue -2.

**Id = IdentityMatrix[5]**

$$\left\{ \{1, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0\}, \{0, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 1\} \right\}$$

**stabspace = NullSpace[MatrixPower[A + 2 Id, 2]]**

$$\left\{ \{2, -2, -2, 0, 1\}, \{-1, 1, 1, 1, 0\} \right\}$$

Thus the stable manifold is spanned by the two vectors

**stab1 = stabspace[[1]]**

$$\{2, -2, -2, 0, 1\}$$

**stab2 = stabspace[[2]]**

$$\{-1, 1, 1, 1, 0\}$$

The original eigenvector must be in this space. We use this to verify our work so far.

**orig = {-1/2, 1/2, 1/2, 5/2, 1};**

**Solve[C1 \* stab1 + C2 \* stab2 == orig, {C1, C2}]**

$$\left\{ \left\{ C1 \rightarrow 1, C2 \rightarrow \frac{5}{2} \right\} \right\}$$

**1 \* stab1 + (5/2) \* stab2**

$$\left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 1 \right\}$$

Everything checks. The most general initial condition which will decay to zero is the most general vector in this manifold, which is given by the expression below for arbitrary C1 and C2.

**C1 \* stab1 + C2 \* stab2**

$$\{2 C1 - C2, -2 C1 + C2, -2 C1 + C2, C2, C1\}$$

The unstable manifold is associated with the eigenvalue 1, and it is spanned by the generalized eigenvectors

tors associated with that eigenvalue.

```
unstabspace = NullSpace[MatrixPower[A - Id, 3]]
{{-2, 1, 0, 0, 1}, {1, 0, 0, 1, 0}, {2, -1, 1, 0, 0}}
```

Once again we check our work by verifying that the original eigenvector is in the space spanned by these.

```
unstab1 = unstabspace[[1]]
{-2, 1, 0, 0, 1}

unstab2 = unstabspace[[2]]
{1, 0, 0, 1, 0}

unstab3 = unstabspace[[3]]
{2, -1, 1, 0, 0}

orig = {0, 1, 0, 2, 1};
Solve[C1 * unstab1 + C2 * unstab2 + C3 * unstab3 == orig, {C1, C2, C3}]
{{C1 -> 1, C2 -> 2, C3 -> 0}}

1 * unstab1 + 2 * unstab2
{0, 1, 0, 2, 1}
```

## ■ Problem 4

We define the system for DynPac.

```
setstate[{x, y, z}]; setparm[{}]; sysname = "Problem 4";
slopevec = {-y + x z - x^4, x + y z + x y z, -z - x^2 - y^2 + z^2 + Sin[x^3]};
```

We find the eigenvalues at the origin.

```
eigsys[{0, 0, 0}]
{{-1, i, -i}, {{0, 0, 1}, {i, 1, 0}, {-i, 1, 0}}}
```

We have a complex conjugate pair, so the origin is a non-hyperbolic equilibrium, and its stability is not determined from the linearization. We see from the eigenvalues that there is a stable manifold of dimension 1 and a center manifold of dimension 2. We now obtain a local approximation for the center manifold, which in turn will lead to a stability determination.

We see from the eigenvalues that the linear stable manifold is the  $z$ -axis, and the linear center manifold is the  $x$ - $y$  plane. The nonlinear center manifold will have the form

$$z = h(x, y) \text{ with } h(0,0) = 0, h_x(0,0) = 0, \text{ and } h_y(0,0) = 0. \quad (1)$$

We differentiate this equation with respect to time to get

$$\dot{z} = h_x \dot{x} + h_y \dot{y}. \quad (2)$$

When the expressions for the time derivatives of  $x$ ,  $y$ , and  $z$  are entered into (2), with  $z$  replaced by  $h$ , we get the basic partial differential equation satisfied by  $h$ . We enter this equation into *Mathematica*. We define first a modification of the slope vector in which  $z$  is replaced by  $h$ .

`ClearAll[h]`

`slopevec`

$$\{-x^4 - y + x z, x + y z + x y z, -x^2 - y^2 - z + z^2 + \text{Sin}[x^3]\}$$

`dotvec = slopevec /. z -> h`

$$\{h x - x^4 - y, x + h y + h x y, -h + h^2 - x^2 - y^2 + \text{Sin}[x^3]\}$$

Because we are going to keep only through cubic terms, we replace  $\text{Sin}[x^3]$  with  $x^3$ .

`dotvec = dotvec /. Sin[x^3] -> x^3`

$$\{h x - x^4 - y, x + h y + h x y, -h + h^2 - x^2 + x^3 - y^2\}$$

Now we expand  $h$  in a Taylor series, keeping terms through order 3. Note that the constant and linear terms are absent because of the conditions in (1).

$$h = a x^2 + 2 b x y + c y^2 + d x^3 + 3 f x^2 y + 3 g x y^2 + k y^3$$

$$a x^2 + d x^3 + 2 b x y + 3 f x^2 y + c y^2 + 3 g x y^2 + k y^3$$

Now we write  $\dot{z} - h_x \dot{x} - h_y \dot{y}$  for *Mathematica*.

`hequat = dotvec[[3]] - D[h, x] dotvec[[1]] - D[h, y] dotvec[[2]]`

$$\begin{aligned} & -x^2 - a x^2 + x^3 - d x^3 - 2 b x y - 3 f x^2 y - y^2 - c y^2 - 3 g x y^2 - \\ & k y^3 + (a x^2 + d x^3 + 2 b x y + 3 f x^2 y + c y^2 + 3 g x y^2 + k y^3)^2 - \\ & (2 a x + 3 d x^2 + 2 b y + 6 f x y + 3 g y^2) \\ & (-x^4 - y + x (a x^2 + d x^3 + 2 b x y + 3 f x^2 y + c y^2 + 3 g x y^2 + k y^3)) - \\ & (2 b x + 3 f x^2 + 2 c y + 6 g x y + 3 k y^2) \\ & (x + y (a x^2 + d x^3 + 2 b x y + 3 f x^2 y + c y^2 + 3 g x y^2 + k y^3) + \\ & x y (a x^2 + d x^3 + 2 b x y + 3 f x^2 y + c y^2 + 3 g x y^2 + k y^3)) \end{aligned}$$

This quantity must vanish. It is a power series in  $x$  and  $y$ , so terms of any given order must vanish separately.

We impose these conditions, starting with the quadratic terms.

$$\text{coeffxx} = (1/2) \text{D}[\text{hequat}, \{x, 2\}] /. \text{Thread}[\{x, y\} \rightarrow \{0, 0\}]$$

$$\frac{1}{2} (-2 - 2a - 4b)$$

$$\text{coeffyy} = (1/2) \text{D}[\text{hequat}, \{y, 2\}] /. \text{Thread}[\{x, y\} \rightarrow \{0, 0\}]$$

$$\frac{1}{2} (-2 + 4b - 2c)$$

$$\text{coeffxy} = \text{D}[\text{D}[\text{hequat}, x], y] /. \text{Thread}[\{x, y\} \rightarrow \{0, 0\}]$$

$$2a - 2b - 2c$$

We solve these for a,b,c.

$$\text{quadans} = \text{Solve}[\{\text{coeffxx} == 0, \text{coeffxy} == 0, \text{coeffyy} == 0\}, \{a, b, c\}]$$

$$\{\{a \rightarrow -1, b \rightarrow 0, c \rightarrow -1\}\}$$

Thus the quadratic approximation to the center manifold is

$$\text{hquad} = -x^2 - y^2$$

$$-x^2 - y^2$$

Now consider the orbits on the center manifold. We have

$$\text{xdot} = \text{slopevec}[[1]] /. z \rightarrow \text{hquad}$$

$$-x^4 - y + x(-x^2 - y^2)$$

$$\text{ydot} = \text{slopevec}[[2]] /. z \rightarrow \text{hquad}$$

$$x + y(-x^2 - y^2) + x y(-x^2 - y^2)$$

Correct to third order, we have the following differential equations on the center manifold:

$$\frac{dx}{dt} = -y - x(x^2 + y^2) + \text{higher order}, \quad \frac{dy}{dt} = x - y(x^2 + y^2) + \text{higher order}. \quad (3)$$

We convert to polar coordinates to get

$$r\dot{r} = -r^4 + \text{higher order}. \quad (4)$$

Equation (4) shows the stability of the equilibrium at the origin.

Had there been no conclusion from this analysis, we would have had to return to the calculation of the center manifold and keep more terms.