Problem 1

Part a

We look for equilibria. The first equation requires \( y = 0 \), and the second equation then requires \( 1 + x^2 = 0 \), so there are no equilibria. Because every periodic solution encloses at least one equilibrium, there can be no periodic solutions.

Part b

The divergence of the slope vector is

\[
3(x - 1)^2 + y^2 + 1 + 3y^2 > 0
\]

Because the divergence doesn't change sign, there can be no periodic solutions, by the Bendixson test.

Alternatively, we note that the only equilibrium is at \((1,0)\), so that any periodic solution must enclose this point. But the \(x\)-axis is an orbit \((\dot{y} = 0 \text{ on } y = 0)\), and so cannot be crossed by another orbit.

Part c

The only equilibrium is \( x = 1, y = 0 \), so any periodic solution must enclose this point. However, the \(x\)-axis is an orbit \((\dot{y} = 0 \text{ on } y = 0)\), so it cannot be crossed by another solution. Hence there are no periodic solutions.

Problem 2

Part a

The linearized equations are \( \dot{x} = x + y \), and \( \dot{y} = -x/2 + 2y \). The eigenvalues for these linear equations are \( \frac{3}{2} \pm \frac{1}{2}i \), so the equilibrium is an unstable spiral.

Part b

We start by calculating \( \dot{V} \).
\[ \dot{V} = 2x\dot{x} + 4y\dot{y} = 2x(x + y - x^3 - 6xy^2) + 4y(-x/2 + 2y - 8y^3 - x^2 y) \]

\[ = 2(x^2 + 4y^2) - 2(x^4 + 8x^2 y^2 + 16y^4) \]

\[ = 2(x^2 + 4y^2)(1 - x^2 - 4y^2). \]

Thus the sign of \( \dot{V} \) is the same as the sign of \( 1 - x^2 - 4y^2 \). On a curve \( V = C \), we may substitute either \( x^2 = C - 2y^2 \) or \( y^2 = \frac{1}{2}(C - x^2) \). Thus on \( V = C \), we get

\[ \dot{V} = 2(x^2 + 4y^2)(1 + x^2 - 2C) = 2(x^2 + 4y^2)(1 - C - 2y^2). \]

On \( V = 0.5 \), the first of these expressions is positive, and on \( V = 1 \), the second expression is negative. Thus the region between \( V = 0.5 \) and \( V = 1.0 \) is a region of orbit trapping. Because there is no equilibrium point in this region (see below for calculations to support this statement), the Poincare-Bendixson Theorem tells us that there is a periodic solution there.

### Part c

Before finding and plotting the limit cycle, we plot the two trapping curves. We define \( \text{trap}[C] \) as the level curve for \( V = C \), and then plot the two curves.

```math
\text{trap}[c_] := \text{ParametricPlot}[[\text{Sqrt}[c] \cos[t], \text{Sqrt}[c/2] \sin[t]], \{t, 0, 2\pi\},
\text{PlotRange} \rightarrow \{\{-1.1, 1.1\}, \{-1.1, 1.1\}\}, \text{PlotStyle} \rightarrow \text{RGBColor}[1, 0, 0]]
```

```math
\text{gr1} = \text{trap}[0.5];
\text{gr2} = \text{trap}[1.0];
\text{sysname} = "Problem 2";
```
show[gr1, gr2]

Problem 2

Now the limit cycle.

setstate[{x, y}]; setparm[{}];
slopevec = {x + y - x^3 - 6 x y^2, -x/2 + 2 y - 8 y^3 - x^2 y};

sysname = "Problem 2";
initvec = {0.75, 0}; t0 = 0.0; h = 0.01; nsteps = 1600;
arrowflag = True; arrowvec = {1/4};
sol2 = limcyc[initvec, t0, h, nsteps];
setcolor[Blue];
Problem 2

We may use the DynPac built-in functions to check the trapping -- as an alternative to our hand calculations in part b. We start by redefining the family of trapping curves. In DynPac, a curve is defined as chain of directed arcs. In this case, a single arc will suffice.
\[ \text{traparc}[c_] := \{\text{Sqrt}[c] \cos[u], \text{Sqrt}[c/2] \sin[u]\}, \{u, 0, 2\pi\} \]

\[ \text{curve}[c_] := \{\text{traparc}[c]\} \]

Now we check these curves for orbit trapping with the function \texttt{orbcross}.

\[ \text{orbcross[\text{curve}[1/2]]} \]

Tangencies: Yes.

Crossings in negative N direction: No.

Crossings in positive N direction: Yes.

This is a trapping curve.

\[ \text{orbcross[\text{curve}[1.0]]} \]

Tangencies: Yes.

Crossings in negative N direction: Yes.

Crossings in positive N direction: No.

This one works. The curve is an outer trapping curve.

We can also use DynPac to apply the Bendixson test. The divergence of the slope vector, with all parameter values substituted is

\[ \text{bendix} = \text{dival[slopevec]} \]

\[ 3 - 4 x^2 - 30 y^2 \]

\[ \text{plrange} = \{\{-1.1, 1.1\}, \{-1.1, 1.1\}\}; \]

We can look for sign changes graphically with the function \texttt{signcontour}. 

The white region corresponds to positive values, the grey to negative. Now we combine this with the earlier graphs.
We see that the limit cycle encloses regions of both signs of the divergence, as it must.

Now we show that there are no equilibria in the trapping region. As we have already found the limit cycle, this is not really necessary, but it is something we would normally do at the beginning of the analysis.
Thus all of the equilibria other than the one at the origin are complex valued. This is not a rigorous proof, because findpolyeq might have missed a root. By making use of the equation for $\dot{V}$, we can prove that the only equilibrium is at the origin. We have $\dot{V} = 2(x^2+4y^2) (1-x^2 - 4y^2)$. At an equilibrium point, $\dot{V}$ must vanish (all orbital derivatives vanish at an equilibrium). Thus any equilibria not at the origin must be on the ellipse $x^2+4y^2=1$. We now set the slope components equal to zero. We then substitute $x^2 = 1 - 4y^2$ into the x-slope to get $y - 2xy^2 = 0$. The choice $y = 0$ leads back to the {0,0} equilibrium. For $y \neq 0$, we get $2xy = 1$. We substitute $y = 1/2x$ into $x^2+4y^2=1$ to get $x^2 + 1/x^2 = 1$, the only roots of which are complex.

**Problem 3**

We derive an equation for $\dot{r}$.

$$ r\dot{r} = x\dot{x} + y\dot{y} = r^2 \sin(r) . $$
It is obvious from this form that there are infinite many limit cycles: namely at \( r = n\pi \). To check the stability, we let \( r = n\pi + \xi \), where \( \xi \) is small. The linearized version of the above equation is

\[
\dot{\xi} = n\pi \cos(n\pi) \xi = (-1)^n n\pi \xi.
\]

For stability, \( \dot{\xi} \) and \( \xi \) must have the opposite sign. Hence the cycles are stable for odd \( n \), unstable for even \( n \).

Let's look at a phase portrait of this unusual system.

```math
\[
\begin{align*}
r &= \sqrt{x^2 + y^2}; \text{setState}[\{x, y\}]; \text{setparm[]}; \\
slopevec &= \{-y + x \sin[r], x + y \sin[r]\}; \text{sysname} = "\text{Problem 3}"; \\
\text{initset} &= \{(2, 0), (5, 0), (7, 0), (-2, 0), (-5, 0), (-7, 0), (11, 0), (-11, 0)\}; \\
t0 &= 0.0; h = 0.02; \text{nsteps} = 300; \\
\text{arrowflag} &= \text{True}; \text{arrowvec} = \{1/8\}; \text{plrange} = \{(-13, 13), (-13, 13)\}; \\
\text{setcolor} &= \{\text{Blue}\};
\end{align*}
\]
Problem 3

We don't see the unstable limit cycle, because no solutions tend toward it. We can see it if we integrate backwards in time.

\[ h = -0.02; \]

\[ \text{setcolor}[(\text{Red})]; \]
Problem 3

We combine the two graphs.
Problem 3

Now we see clearly the alternating stability of the limit cycles.

Problem 4

We derive an equation for $\dot{r}$.

$$r\dot{r} = x\dot{x} + y\dot{y} = r^2(1 - r^2)^2.$$  

We see that $\dot{r}$ is always positive except at $r = 0$ or $1$ where it is zero. The equilibrium point is $r = 0$, and it is shown easily to be an unstable spiral. Clearly $r = 1$ is a limit cycle. For $r < 1$, the system point approaches the limit cycle because $\dot{r}$ is positive. For $r > 1$, $\dot{r}$ is still positive, and system points move away from the limit cycle. Thus the cycle looks stable from the inside, but unstable from the outside. The net result is that it is unstable, because there are perturbations that will not come back. Sometimes such a cycle is called semi-stable. We take a quick look at the orbits.
setstate[{x, y}]; setparm[{}];
slopevec = \{-y + x \left(1 - x^2 - y^2\right)^2, x + y \left(1 - x^2 - y^2\right)^2\};
sysname = "Problem 4";
initset = {{0.1, 0}, {-0.1, 0}, {1.1, 0}, {-1.1, 0}};
t0 = 0.0; h = 0.02; nsteps = 500;
arrowflag = True; arrowvec = {1/8, 7/8};
setcolor[{Blue, Blue, Red, Red}];
plrange = {{-2, 2}, {-2, 2}};
rangeflag = True; ranger = plrange;
bothdirflag = True;
gr8 = portrait[initset, t0, h, nsteps, 1, 2]

Problem 4
**Problem 5**

In the notation of the theorem, \( f(x) = x^2 - 3 \) and \( g(x) = (x + x^3)/(1 + x^2) \). We check the hypotheses of the theorem as given in class: (i) both \( f \) and \( g \) are continuously differentiable; (ii) \( g \) is an odd function; (iii) \( g \) is positive for \( x \) positive; (iv) \( f \) is an even function; (v) \( F(x) = \int_0^x f(u) \, du = \frac{1}{3}x^3 - 3x \), \( F \) has exactly one positive zero, at \( x = 3 \), \( F \) is negative for \( 0 < x < 3 \), and \( F \) is positive and nondecreasing for \( x > 3 \). Therefore the theorem applies, and it guarantees that there is a unique stable limit cycle. We construct the cycle now.

```math
setstate[{x, y}]; setparm[{}];
slopevec = {y, (3 - x^2) y - (x + x^3)/(1 + x^2)}; sysname = "Problem 5";
classify2D[{0, 0}]
```

Abbreviations used in classify2D.

\( L = \) linear, \( NL = \) nonlinear, \( R2 = \) repeated root.

\( Z1 = \) one zero root, \( Z2 = \) two zero roots.

This message printed once.

unstable - node

We start with one initial condition.

```math
init = {1, 0};
t0 = 0.0; h = 0.05; nsteps = 500; plrange = {{-10, 10}, {-10, 10}};
arrowflag = True; arrowvec = {1/2};
bothdirflag = True;
rangeflag = False;
sol5 = integrate[init, t0, h, nsteps];
```
Problem 5

Now we construct the pure limit cycle.

\[
\text{bothdirflag} = \text{False};
\]
\[
\text{t0} = 0.0; \text{h} = 0.02; \text{nsteps} = 500;
\]
\[
\text{sol6} = \text{limcyc}[[3, 0], \text{t0}, \text{h}, \text{nsteps}];
\]
\[
\text{period}[\text{sol6}]
\]
\[
8.86
\]
This looks a lot like the van der Pol oscillator, as we might expect from the form of the equation. This suggests an alternative approach to the problem -- namely rescaling the original equation to produce the van der Pol equation. We start by noting that \( g(x) = (x + x^3)/(1 + x^2) = x \). Then the equation is \( \ddot{x} + (x^2 - 3)x' + x = 0 \). This suggests we try \( z = \sqrt{3} x \). That substitution gives \( \ddot{z} + 3(z^2 - 1)\dot{z} + z = 0 \), which is the van der Pol equation with \( \mu = 3 \).

Problem 6

We will compute a few orbits before speculating.

```
setstate[{x, y}]; setparm[{}]; slopevec = {y, -x y - x^3};
sysname = "Problem 6";
```
\[
t_0 = 0.0; \ h = 0.04; \ nsteps = 1000;
\]
bothdirflag = False; rangeflag = False;
initset = \{(0, 0.1), (0, 0.5), (0, 0.75), (0, 1), (0, 1.5)\};
ardownflag = True; arrowvec = \{1/2\};
plrange = \{(-1.51, 1.51), (-1.51, 1.51)\};
setcolor[\{Blue\}];
gr11 = portrait[initset, t0, h, nsteps, 1, 2]

On the basis of this surprising graph, we would speculate that all of the solutions of this equation are periodic. In that case we would have an example of a dissipative nonlinear center. The correctness of this speculation is shown in Jordan and Smith, 4th edition, example 11.5, section 11.2.