

ME 406

Assignment #4 Solutions

`sysid`

Mathematica 6.0.3, DynPac 11.01, 1/30/2009

■ Problem 1

■ Part a

The right-hand sides at the origin are $\sin(0+0) = 0$ and $\sin(0-0) = 0$, so $(0,0)$ is an equilibrium. The right-hand sides are $\sin(x+y) = x+y + (\text{higher order terms})$, and $\sin(x-y) = x-y + (\text{higher order terms})$. Thus the linearized equations are

$$\dot{x} = x + y, \quad \dot{y} = x - y. \quad (1)$$

We try $x = ue^{rt}$ and $y = ve^{rt}$ to get

$$(1-r)u + v = 0, \quad \text{and} \quad u - (1+r)v = 0. \quad (2)$$

We require the determinant to vanish for a non-trivial solution and this gives $r^2 = 2$, hence $r = \pm\sqrt{2}$. Thus the equilibrium is a saddle and is unstable both for the linearized and exact equations. We check this with DynPac.

```
setstate[{x, y}]; setparm[{}]; slopevec = {Sin[x+y], Sin[x-y]};
```

```
classify2D[{0, 0}]
```

Abbreviations used in classify2D.

L = linear, NL = nonlinear, R2 = repeated root.

Z1 = one zero root, Z2 = two zero roots.

This message printed once.

unstable - saddle

```
eigval[{0, 0}]
```

```
{-sqrt(2), sqrt(2)}
```

■ Part b

The right-hand sides at the origin are $e^0 - \cos(0) + \sinh(0) = 1 - 1 + 0 = 0$, and $e^0 - \cos(0) - 0 = 1 - 1 - 0 = 0$, so $(0,0)$ is an equilibrium. By using Taylor expansions for the elementary functions, we get for the right-hand side of the x -equation $(1 + x + x^2/2 + \dots) - (1 - x^2/2 + \dots) + (y + y^3/6 + \dots) = x + y + (\text{higher order terms})$. The right-hand side of the y -equation is $(1 - y + y^2/2 \dots) - (1 - x^2/2 + \dots) - y = -2y + (\text{higher order terms})$. Thus the linearized equations are

$$\dot{x} = x + y, \quad \dot{y} = -2y \quad . \quad (3)$$

We try $x = ue^{rt}$ and $y = ve^{rt}$ to get

$$(1 - r)u + v = 0, \quad \text{and} \quad -(2 + r)v = 0. \quad (4)$$

We require the determinant to vanish for a non-trivial solution and this gives $r = -2, 1$. Thus the equilibrium is a saddle and is unstable both for the linearized and exact equations. We check this with DynPac.

```
slopevec = {Exp[x] - Cos[x] + Sinh[y], Exp[-y] - Cos[x] - y};
classify2D[{0, 0}]
unstable - saddle
eigval[{0, 0}]
{-2, 1}
```

■ Part c

The right-hand sides at the origin are $0/(1 + 0) = 0$, and $e^0 - 1 = 1 - 1 = 0$, so $(0,0)$ is an equilibrium. For variety, we calculate the derivative matrix in this problem rather than calculating the Taylor series directly. We have

$$f(x,y) = y/(1 + x^2), \quad g(x,y) = e^{-x} - 1, \quad ,$$

so that the partial derivatives evaluated at the origin are

$$\partial_x f = 0, \quad \partial_y f = 1, \quad \partial_x g = -1, \quad \partial_y g = 0.$$

Then the linearized equations are

$$\dot{x} = y, \quad \dot{y} = -x \quad . \quad (5)$$

We try $x = ue^{rt}$ and $y = ve^{rt}$ to get

$$-ru + v = 0, \quad \text{and} \quad -u - rv = 0. \quad (6)$$

We require the determinant to vanish for a non-trivial solution and this gives $r = \pm i$. Thus the equilibrium is a center for the linear system and hence there is no conclusion for the exact equations. We check this with DynPac.

```
slopevec = {y / (1 + x^2), Exp[-x] - 1};
```

```
classify2D[{0, 0}]
```

```
stable (L), indeterminate (NL) - center
```

```
eigval[{0, 0}]
```

```
{i, -i}
```

■ Part d

The right-hand sides at the origin are $\tan(0+0) = 0$, and $-(12)0 - 6 \sin(0) = 0$, so $(0,0)$ is an equilibrium. Here also we calculate the derivative matrix in this problem rather than calculating the Taylor series directly. We have

$$f(x,y) = \tan(x + y), \quad g(x,y) = -12x - 6\sin(y) ,$$

so that the partial derivatives evaluated at the origin are

$$\partial_x f \partial_x f \partial_x f = 1, \quad \partial_y f = 1, \quad \partial_x g = -12, \quad \partial_y g = -6. \partial_x f$$

Then the linearized equations are

$$\dot{x} = x + y, \quad \dot{y} = -12x - 6y . \quad (7)$$

We try $x = ue^{rt}$ and $y = ve^{rt}$ to get

$$(1 - r)u + v = 0, \quad \text{and} \quad -12u - (r + 6)v = 0. \quad (8)$$

We require the determinant to vanish for a non-trivial solution and this gives $r = -2, -3$. Thus the equilibrium is a strictly stable node for both the linear system and the exact system. We check this with DynPac.

```
slopevec = {Tan[x + y], -12 x - 6 Sin[y]};
```

```
classify2D[{0, 0}]
```

```
strictly stable - node
```

```
eigval[{0, 0}]
```

```
{-3, -2}
```

■ Problem 2

■ Part a

The right-hand sides obviously vanish at the origin, so the origin is an equilibrium. There may be others, but we are not asked to consider them in this problem. We get the linearized equations by dropping all of the higher order terms:

$$\dot{x} = -y, \quad \dot{y} = 4x . \quad (9)$$

You may recognize these as the equations of an undamped oscillator, in which case you know already that the linearization is not going to lead to any conclusion about the stability. If you don't recognize this, you will soon find out: the eigenvalues are easily calculated as $\pm 2i$. We check this conclusion with DynPac.

```
slopevec = {-y - 1/2 x^3 + 1/2 x^4 + 3/2 y^4 - 3/4 y^5, 4x - 6y^3 + x^4 + 3y^4 - x^5};
classify2D[{0, 0}]
stable (L), indeterminate (NL) - center
eigval[{0, 0}]
{2 i, -2 i}
```

Just out of curiosity, we see if there are any other equilibrium points by using `nfindpolyeq`.

```
nfindpolyeq
{{-1.21733, 2.}, {-0.383225 - 1.76687 i, -0.743413 + 1.15521 i},
{-0.383225 + 1.76687 i, -0.743413 - 1.15521 i},
{0.383225 - 1.76687 i, 0.743413 + 1.15521 i},
{0.383225 + 1.76687 i, 0.743413 - 1.15521 i},
{0.234465 - 1.3507 i, 2.}, {0.234465 + 1.3507 i, 2.},
{-1.08109 - 0.234482 i, 0.454868 + 0.706835 i},
{-1.08109 + 0.234482 i, 0.454868 - 0.706835 i},
{1.08109 - 0.234482 i, -0.454868 + 0.706835 i},
{1.08109 + 0.234482 i, -0.454868 - 0.706835 i},
{1., -0.462164 + 0.659993 i}, {1., -0.462164 - 0.659993 i},
{1.76687 - 0.383225 i, 1.15521 + 0.743413 i},
{1.76687 + 0.383225 i, 1.15521 - 0.743413 i},
{-1.76687 - 0.383225 i, -1.15521 + 0.743413 i},
{-1.76687 + 0.383225 i, -1.15521 - 0.743413 i},
{0.234482 - 1.08109 i, 0.706835 - 0.454868 i},
{0.234482 + 1.08109 i, 0.706835 + 0.454868 i},
{-0.234482 - 1.08109 i, -0.706835 - 0.454868 i},
{-0.234482 + 1.08109 i, -0.706835 + 0.454868 i},
{1.7484, 2.}, {1., 1.75211}, {1., 1.17221}, {0., 0.}}
```

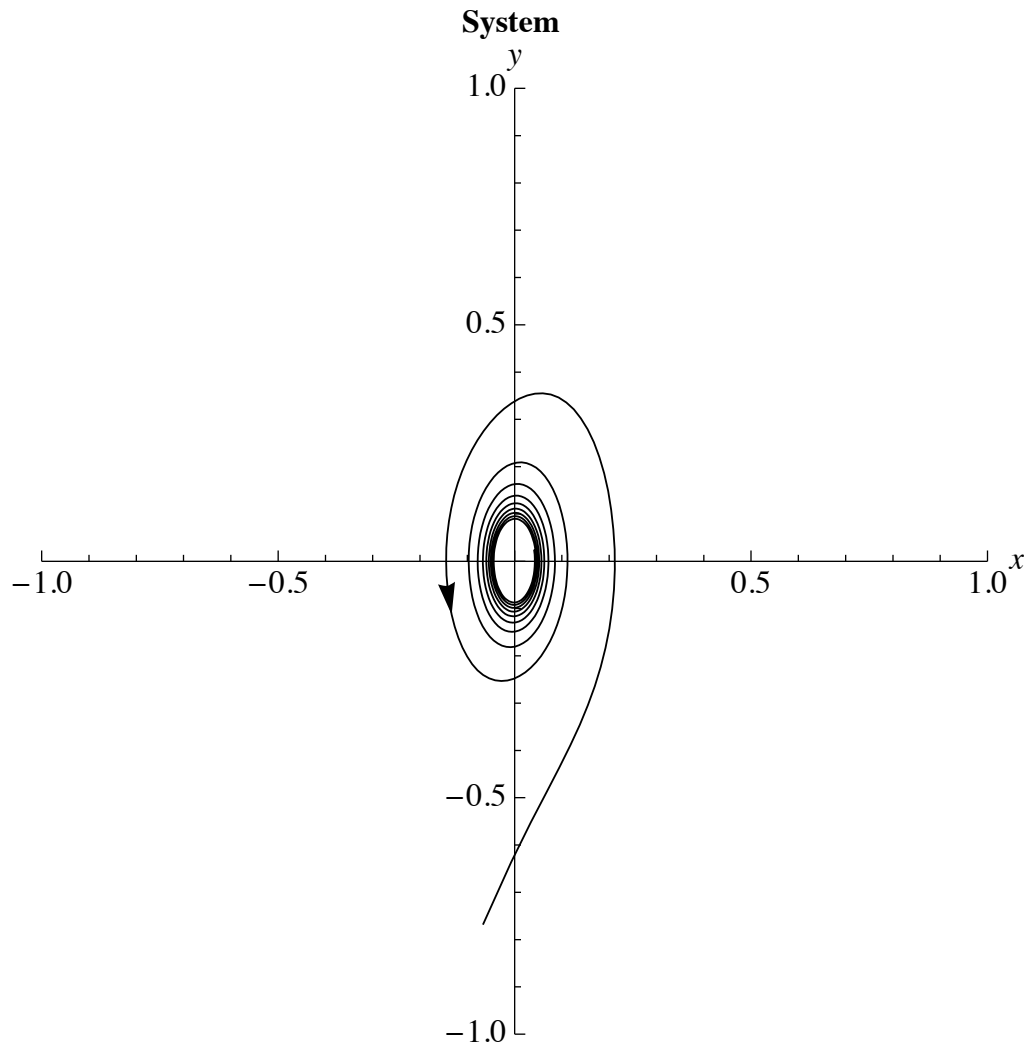
We see that there are (at least) four other real equilibria. Such a system is likely to have a complex and interesting phase portrait.

■ Part b

We now look at an orbit in the neighborhood of the origin to get some idea about the stability. We integrate both directions in time, and we turn on range checking to prevent blowups. We start quite near the origin.

```
initvec = {0.1, 0.1}; t0 = 0.0; h = 0.05; nsteps = 500;
plrange = {{-1, 1}, {-1, 1}}; ranger = plrange;
```

```
rangeflag = True; bothdirflag = True;  
sol1 = integrate[initvec, t0, h, nsteps];  
arrowflag = True; arrowvec = {1 / 4};  
phaser[sol1]
```



We see that the orbit appears to be spiralling in to the origin. The process is slow, because it is the higher order nonlinear terms which drive the system point inward, and they are very small near the origin. The orbits for the linearized equations are ellipses.

■ Problem 3

■ Part a

We attack this question by asking what happens on the two boundaries of the first quadrant -- that is, on the lines $S = 0$ and $Y = 0$. On the line $S = 0$, we see from the S -equation that $\dot{S} = \Gamma > 0$, so the system point cannot cross this line from the positive S side. On the line $Y = 0$, we find from the Y -equation that $\dot{Y} = 0$, so once again the system point cannot cross this line, although this shows that it will stay on the line once it gets there.

■ Part b

We define the system for DynPac and then look for equilibria.

```
setstate[{S, Y}]; setparm[{alpha, delta, Gamma, r}];
slopevec = {Gamma - delta S - alpha S Y, alpha S Y - r Y - delta Y};

sysname = "Epidemic";

eqs = findpolyeq
```

$$\left\{ \left\{ \frac{\Gamma}{\delta}, 0 \right\}, \left\{ \frac{r + \delta}{\alpha}, \frac{\Gamma - \frac{\delta(r + \delta)}{\alpha}}{r + \delta} \right\} \right\}$$

Thus we get two equilibria. In the first, there are no infectives, and the population of susceptibles is a result of a balance between death and addition of new susceptibles. In the second equilibrium, there are both susceptibles and infectives present. We name our two equilibria:

```
eqfree = eqs[[1]]
```

$$\left\{ \frac{\Gamma}{\delta}, 0 \right\}$$

```
eqend = eqs[[2]]
```

$$\left\{ \frac{r + \delta}{\alpha}, \frac{\Gamma - \frac{\delta(r + \delta)}{\alpha}}{r + \delta} \right\}$$

It is easy to show directly from the equations that there are no other equilibria.

■ Part c

From the expression for eqend, we see that it is relevant for $\Gamma > \delta(r + \delta)/\alpha$, and irrelevant when the inequality is reversed. Now we look at the stability of the two equilibria. We use the DynPac function eigval to get the eigenvalues. First the disease-free state:

`eigval[eqfree]`

$$\left\{ -r + \frac{\alpha \Gamma}{\delta} - \delta, -\delta \right\}$$

The second eigenvalue $-\delta$ is always negative. The stability is then determined by the first eigenvalue, which may be written as

$$[\alpha/\delta][\Gamma - \delta(r + \delta)/\alpha] .$$

In this form it is obvious that the equilibrium is stable for $\Gamma < \delta(r + \delta)/\alpha$ and unstable for $\Gamma > \delta(r + \delta)/\alpha$.

We have to work a little harder for the second equilibrium. The eigenvalues there are

`Simplify[eigval[eqend]]`

$$\left\{ -\frac{\alpha \Gamma + \sqrt{\alpha^2 \Gamma^2 + 4 (r + \delta)^2 (-\alpha \Gamma + \delta (r + \delta))}}{2 (r + \delta)}, \right. \\ \left. \frac{-\alpha \Gamma + \sqrt{\alpha^2 \Gamma^2 + 4 (r + \delta)^2 (-\alpha \Gamma + \delta (r + \delta))}}{2 (r + \delta)} \right\}$$

From this form we can see that if $\alpha\Gamma < \delta(r + \delta)$, then the quantity under the square root sign is positive and greater than $\alpha\Gamma$ in magnitude. Thus the second eigenvalue will be positive and the equilibrium is unstable. If on the other hand $\alpha\Gamma > \delta(r + \delta)$, then either both roots are real and negative, or are complex conjugate with a negative real part. The result of this analysis may be summarized simply as follows: For $\Gamma < \delta(r + \delta)/\alpha$, the only stable equilibrium is the disease-free state in which the population of susceptibles is Γ/δ . If any infectives are introduced into the population, the disease will eventually die out and the disease-free equilibrium will be restored. For $\Gamma > \delta(r + \delta)/\alpha$, the only stable equilibrium is an endemic one. The system may oscillate around this state if perturbed, but it will always return to the endemic state in which the disease is present. If one manages to produce a disease-free equilibrium, it will be unstable, and the introduction of even one infective will drive the system to the endemic state. (Example: Now that smallpox has been eradicated, it is no longer necessary to vaccinate for it. The resulting disease-free state is unstable, and if smallpox is introduced either through a laboratory accident or terrorist attack, there could be a devastating epidemic.)

■ Part d

We set the parameter values, evaluate the equilibrium states, check the stability conclusions and then carry out an integration.

`parmvec`

$$\{\alpha, \delta, \Gamma, r\}$$

$$\text{parmval} = \{5 \cdot 10^{-4}, 0.1, 2000.0, 5.0\};$$

```
eqfree /. Thread[parmvec -> parmval]
```

```
{20 000., 0}
```

```
eqend /. Thread[parmvec -> parmval]
```

```
{10 200., 192.157}
```

```
classify2D[eqfree]
```

```
unstable - saddle
```

```
classify2D[eqend]
```

```
strictly stable - spiral
```

These results are consistent with our above results for the case $\Gamma > \delta(r+\delta)/\alpha$. We verify that the inequality is satisfied.

```
( $\Gamma > \delta(r+\delta)/\alpha$ ) /. Thread[parmvec -> parmval]
```

```
True
```

Now for the integration. First we reset the integration parameters to the default values.

```
intreset;
```

```
initvec = {20 000, 1}; t0 = 0.0; h = 0.02; nsteps = 2000;
```

```
solep = integrate[initvec, t0, h, nsteps];
```

```
plrange = {{0, 40}, {0, 20 000}};
```

```
asprat = 0.7;
```

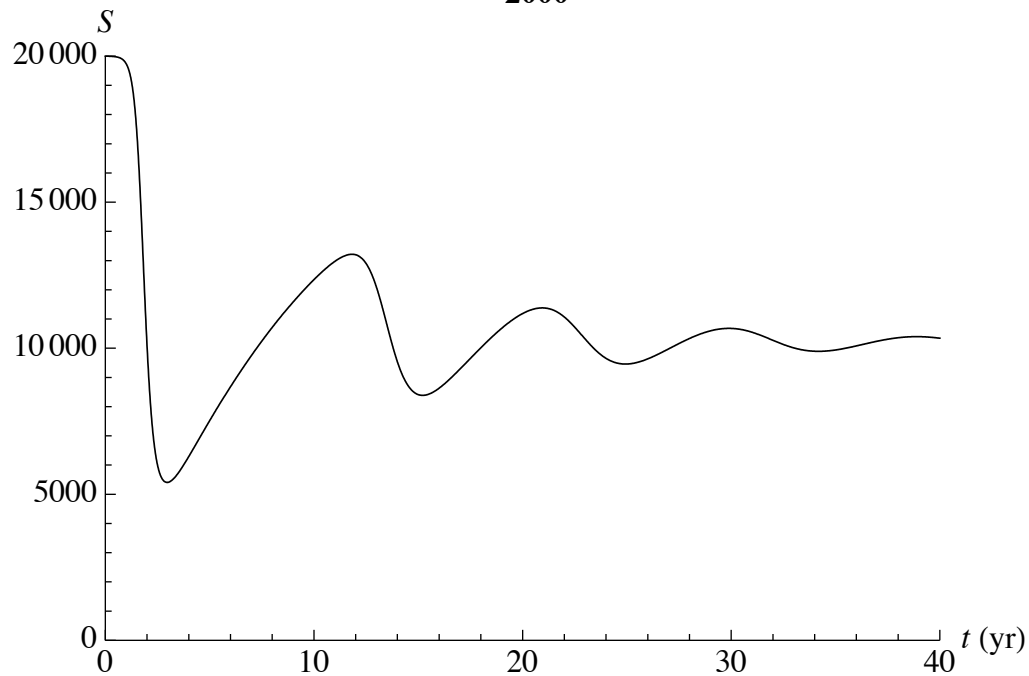
```
timeunit = "yr";
```

```
imsize = 380;
```

```
sciflag = True; decdig = 2;
```

```
graphS = timeplot[solep, 1]
```

$$\text{Epidemic } \{\alpha, \delta, \Gamma, r\} = \left\{ \frac{1}{2000}, 1. \times 10^{-1}, 2. \times 10^3, 5. \right\}$$

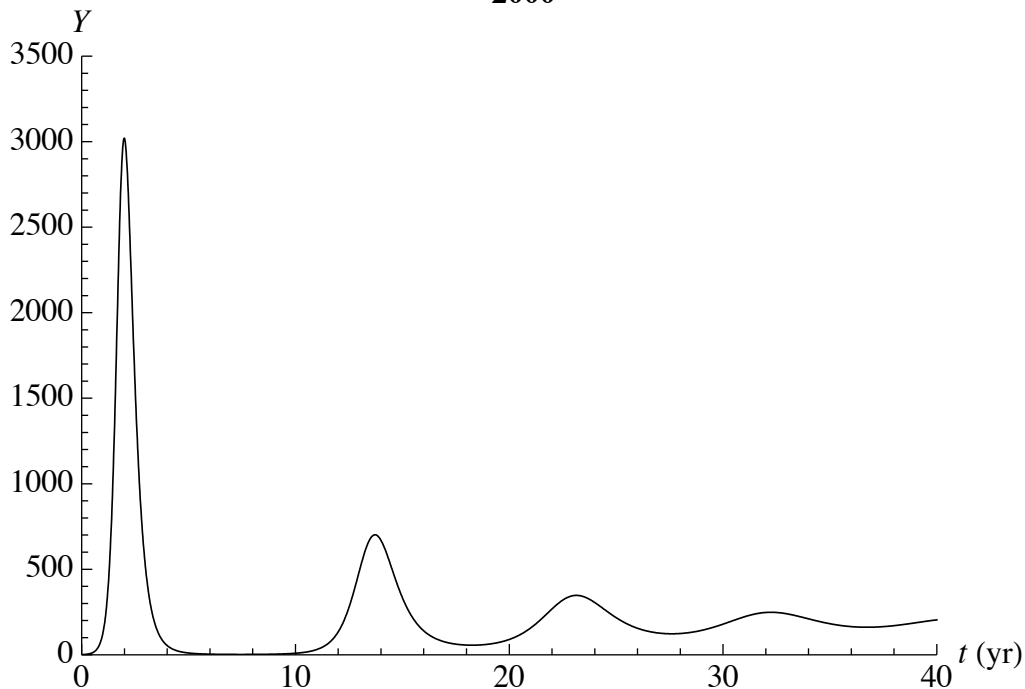


This is a surprising and interesting result. The system exhibits an oscillation which takes about 40 years to damp out. The peaks are very roughly 10 years apart. Let's conclude this problem by plotting the number of infectives.

```
plrange = {{0, 40}, {0, 3500}};
```

```
graphY = timeplot[solep, 2]
```

$$\text{Epidemic } \{\alpha, \delta, \Gamma, r\} = \left\{ \frac{1}{2000}, 1. \times 10^{-1}, 2. \times 10^3, 5. \right\}$$



We use the function `staterange` to find the maximum number of infectives over the epidemic.

```
staterange[solep]
```

```
{{S, {5405.63, 2.96}, {20000, 0.}}, {Y, {1, 0.}, {3019.8, 2.}}}
```

This tells us that the minimum S is 5405.63 at 2.96 years, the maximum S is 20000 at the initial time, the minimum Y is 1 at the initial time, and the maximum Y is 3019.8 at 2 years. So the maximum number of infectives is 3020.

Although this model is oversimplified, oscillations of this sort are sometimes observed with real diseases.