ME 406 ASSIGNMENT #5

PROBLEMS DUE IN CLASS ON THURSDAY FEB. 19, 2009

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
I. PLANE AUTONOMOUS SYSTEMS		
1.5 Stability by Liapunov Methods	Th, T Feb 5,10	
1.6 Predator-Prey Part 1	Th Feb 12	---
1.7 Examples of Periodic Solutions	T Feb 17	

In addition to the text reading, there is some relevant material for this assignment in the DynPac Tutorial 11 on Liapunov methods, in the predator-prey handout, and in the handout on examples of periodic solutions.

PROBLEMS

(1) (25 points) This problem deals with the system given below, which was also considered in problem 2 of Assignment #4. As you showed there, the origin is an equilibrium, and stability analysis by linearization is inconclusive. In this problem you are asked to demonstrate the stability of the equilibrium of the origin by finding a Liapunov function.

$$\dot{x} = -y - \tfrac{1}{2}x^3 + \tfrac{1}{2}x^4 + \tfrac{3}{2}y^4 - \tfrac{3}{4}y^5 \;,\; \dot{y} = 4x - 6y^3 + x^4 + 3y^4 - x^5 \,.$$

(2) (25 points) In 1926, the mathematician Vito Volterra proposed a simple model for describing predator-prey situations. A similar set of equations had been proposed earlier by Alfred Lotka as a model of oscillating chemical reactions. In the notation of our predator-prey example done in class, the equations, now called the Lotka-Volterra equations, have the following form:

$$\frac{dM}{dt} = aM - bMV \;,\; \frac{dV}{dt} = cMV - dV \,,$$

where a, b, c, and d are positive constants. Here M is the prey population and V is the predator population.

(a) Give a simple interpretation of each of the four terms in the equations, in terms of births, deaths, interactions, etc.

(b) By exploring this system with any tools that you like, try to draw some conclusions about the general behavior of solutions of this system. Use $a = 4$, $b = 2$, $c = 3$, and $d = 1$ for the parameter values.

(c) Using your results of parts (a) and (b), comment on the strengths and weaknesses of these equations as a model for predator-prey behavior.

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(3) (25 points) Consider the equation $\ddot{x} + x^7 = 0$.

(a) Derive the energy equation and show that all of the solutions are periodic.

(b) Convert this equation to a system in the usual way by letting $y = dx/dt$. Each orbit can be characterized by where it crosses the positive x-axis -- that is, by the initial conditions $x(0) = x_0$ with $x_0 > 0$, and $y(0) = 0$. Find (any way you like) the period T_1 of the orbit for $x_0 = 1$.

(c) Show that for any other orbit, the period is $T = x_0^{-3} T_1$. (Hint: By scaling both dependent and independent variables, relate the general case to $x_1(t)$.)

(d) Use DynPac to construct a few of the orbits and to find their periods. Compare the periods found with DynPac to the results from the formula of part (c).

(4) (25 points) Consider the system given by

$$\dot{x} = 2x + 2y - x(2x^2 + y^2)\,,\ \dot{y} = -2x + y - y(2x^2 + y^2)\ .$$

(a) Use linear theory to analyze the stability of the equilibrium at the origin.

(b) By finding an equation for dr/dt, show that the system has a limit cycle, and show that all initial conditions tend toward this limit cycle.

(c) Use DynPac to construct the limit cycle, along with some neighboring orbits.

(d) Find the period of the limit cycle.