ME 406 ASSIGNMENT #4

PROBLEMS DUE IN CLASS ON THURSDAY FEB. 12, 2009

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
I. PLANE AUTONOMOUS SYSTEMS		
1.4 Stability of Nonlinear Systems by Linearization	T,Th Feb 3, 5	6.3, 6.4

In this assignment, you will do some calculations by hand and some with DynPac, all relating to the stability of equilibria as determined by linearization. The relevant DynPac tutorials are Tutorial Four on the local solution near an equilibrium point and Tutorial Three on Equilibrium and Stability. You are not required to do the tutorials, but you may find them helpful in doing the homework. Stability by the Liapunov method will be covered on the next assignment.

PROBLEMS

(1) (30 points) For each of the systems below, show that there is an equilibrium point at the origin, and use linearization to determine the stability of the equilibrium. Do the calculations by hand, but check your answer with the DynPac function classify2D.

(a) $\dot{x} = \sin(x+y)$, $\dot{y} = \sin(x-y)$.

(b) $\dot{x} = e^{x} - \cos(x) + \sinh(y)$, $\dot{y} = e^{-y} - \cos(x) - y$.

(c) $\dot{x} = \dfrac{y}{1+x^2}$, $\dot{y} = e^{-x} - 1$.

(d) $\dot{x} = \tan(x+y)$, $\dot{y} = -12x - 6\sin(y)$.

(2) (20 points) This problem deals with the following system:

$$\dot{x} = -y - \tfrac{1}{2}x^3 + \tfrac{1}{2}x^4 + \tfrac{3}{2}y^4 - \tfrac{3}{4}y^5 \ , \dot{y} = 4x - 6y^3 + x^4 + 3y^4 - x^5 \ .$$

(a) Show that the origin is an equilibrium point, and attempt to determine the stability by linearization.

(b) Use DynPac to look at the orbits near the origin, and draw a tentative conclusion about the stability of the equilibrium there.

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(3) (50 points) This problem deals with a simple model for an infectious disease in a given population. We may look at such models in more detail later. In the population, there are at any given time t a total number $S(t)$ individuals susceptible to the disease, and a total number $Y(t)$ who are infected. The model consists of two coupled differential equations governing these populations. The equations are

$$\frac{dS}{dt} = \Gamma - \delta S - \alpha SY \quad , \quad \frac{dY}{dt} = \alpha SY - rY - \delta Y .$$

In the first equation, the terms are as follows: (1) Γ is the number of new susceptibles appearing per unit time, either by births or immigration; (2) δ is the per capita death rate and δS is the total number of deaths of susceptibles per unit time; (3) αSY is the number of new infections per unit time, based on the idea that the encounter rate between susceptibles and infectives is proportional to the product of the two populations, and that the infection rate is proportional to the encounter rate. In the second equation, we have the following terms: (1) αSY is the rate at which new infectives appear; (2) rY is the rate at which infectives recover and become immune to this disease; (3) δY is the number of deaths of infectives per unit time. Although this model is not unreasonable, it is oversimplified. You might want to think about ways in which it could be improved. You may use any methods you like in answering the questions below.

(a) Because the dependent variables are populations, only non-negative values make any sense. Thus we need to remain in the first quadrant of the phase plane. Show that if we start with an initial condition for which both values are non-negative, the state of the system will always remain in the first quadrant of the phase plane. (Hint: do this by showing that the slope vector is such that a phase point in that quadrant can never cross into the neighboring quadrants.)

(b) Show that there are two equilibrium states. One of these states is disease free ($Y = 0$), and the other is an equilibrium in which the disease is present (such a state is called endemic in epidemiology).

(c) We call an equilibrium state relevant if the coordinates of the state are both non-negative. Derive the following results. If $\Gamma < \frac{\delta(r+\delta)}{\alpha}$, then the disease free equilibrium is stable, and the endemic state is both irrelevant and unstable. If $\Gamma > \frac{\delta(r+\delta)}{\alpha}$, the disease free state is unstable, and the endemic state is both relevant and stable. Thus for given rate constants α, δ and r, we will get epidemics and endemic states for $\Gamma > \frac{\delta(r+\delta)}{\alpha}$ – i.e. for a sufficiently large value of Γ. (Useful hint: If the eigenvalues λ satisfy a quadratic equation of the form $\lambda^2 + a\lambda + b = 0$, then the associated equilibrium is stable if and only if both a and b are positive.)

(d) Explore this system with DynPac for the following parameter values:

$$\alpha = 5 \times 10^{-4} \text{ yr}^{-1} , \delta = 0.1 \text{ yr}^{-1} , r = 5 \text{ yr}^{-1} , \Gamma = 2000 \text{ yr}^{-1} .$$

In particular, check your stability conclusions from above, and construct some time plots of the solution. For the initial state, take S equal to the disease-free equilibrium value, and take a very small number of initial infectives (e.g., 1). Find the maximum number of infectives over the course of the epidemic.