ME 406
Example of a Phase Portrait with Multiple Equilibria

```
sysid
```

```
Mathematica  6.0.3, DynPac 11.01, 1/13/2009
```

```
plotreset; intreset; imsize = 250;
```

In this notebook, we construct a phase portrait for the sytem given below.

$$\dot{x} = -y - \frac{1}{2}x^3 + \frac{1}{2}x^4 + \frac{3}{2}y^4 - \frac{3}{4}y^5 \ , \ \dot{y} = 4x - 6y^3 + x^4 + 3y^4 - x^5 \ . \qquad (1)$$

We begin by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{}];
```

$$\text{slopevec} = \left\{-y - \frac{1}{2}x^3 + \frac{1}{2}x^4 + \frac{3}{2}y^4 - \frac{3}{4}y^5,\ 4x - 6y^3 + x^4 + 3y^4 - x^5\right\};$$

```
sysname = "Portrait";
```

We use nfindpolyeq to look for equilibrium states.

```
eqstates = nfindpolyeq
```

```
{{-1.21733, 2.}, {-0.383225 - 1.76687 i, -0.743413 + 1.15521 i},
 {-0.383225 + 1.76687 i, -0.743413 - 1.15521 i},
 {0.383225 - 1.76687 i, 0.743413 + 1.15521 i},
 {0.383225 + 1.76687 i, 0.743413 - 1.15521 i},
 {0.234465 - 1.3507 i, 2.}, {0.234465 + 1.3507 i, 2.},
 {-1.08109 - 0.234482 i, 0.454868 + 0.706835 i},
 {-1.08109 + 0.234482 i, 0.454868 - 0.706835 i},
 {1.08109 - 0.234482 i, -0.454868 + 0.706835 i},
 {1.08109 + 0.234482 i, -0.454868 - 0.706835 i},
 {1., -0.462164 + 0.659993 i}, {1., -0.462164 - 0.659993 i},
 {1.76687 - 0.383225 i, 1.15521 + 0.743413 i},
 {1.76687 + 0.383225 i, 1.15521 - 0.743413 i},
 {-1.76687 - 0.383225 i, -1.15521 + 0.743413 i},
 {-1.76687 + 0.383225 i, -1.15521 - 0.743413 i},
 {0.234482 - 1.08109 i, 0.706835 - 0.454868 i},
 {0.234482 + 1.08109 i, 0.706835 + 0.454868 i},
 {-0.234482 - 1.08109 i, -0.706835 - 0.454868 i},
 {-0.234482 + 1.08109 i, -0.706835 + 0.454868 i},
 {1.7484, 2.}, {1., 1.75211}, {1., 1.17221}, {0., 0.}}
```

```
Length[eqstates]
```

```
25
```

We have found five real equilibria. We name them.

```
eq1 = eqstates[[22]]
```

```
{1.7484, 2.}
```

```
eq2 = eqstates[[1]]
```

```
{-1.21733, 2.}
```

```
eq3 = eqstates[[23]]
```

```
{1., 1.75211}
```

```
eq4 = eqstates[[24]]
```

```
{1., 1.17221}
```

```
eq5 = eqstates[[25]]
```

```
{0., 0.}
```

Now we classify all of these equilibria.

```
classify2D[eq1]
```

```
unstable - saddle
```

```
classify2D[eq2]
```

```
unstable - saddle
```

```
classify2D[eq3]
```

```
unstable - node
```

```
classify2D[eq4]
```

```
unstable - saddle
```

```
classify2D[eq5]
```

```
stable (L), indeterminate (NL) - center
```

We see that four of the equilibria are unstable, and the fifth -- the origin -- has a stability which is not determined by linearization. As we showed earlier, the stability of the equilibrium at the origin is established by a Liapunov function, namely

```
Vlap = 4 x^2 + y^2;
```

The orbital derivative of Vlap is

```
Simplify[orbdt[Vlap]]
```

$-2\ (-1+x)\ (-2+y)\ \left(x^4+3\ y^4\right)$

and this is clearly negative definite in a neighborhood of the origin.

We begin the phase portrait by constructing the stable and unstable manifolds about the saddle points. We set the plotting window:

```
plrange = {{-4.1, 4.1}, {-4.1, 4.1}};
```

We set range checking and integration in only one time direction.

```
rangeflag = True; ranger = plrange; bothdirflag = False;

eig1 = eigsys[eq1]
```

```
{{33.961, -3.85689}, {{0.422889, -0.906182}, {-0.793775, -0.608212}}}
```

```
eigvec11 = First[Last[eig1]]
```

```
{0.422889, -0.906182}
```

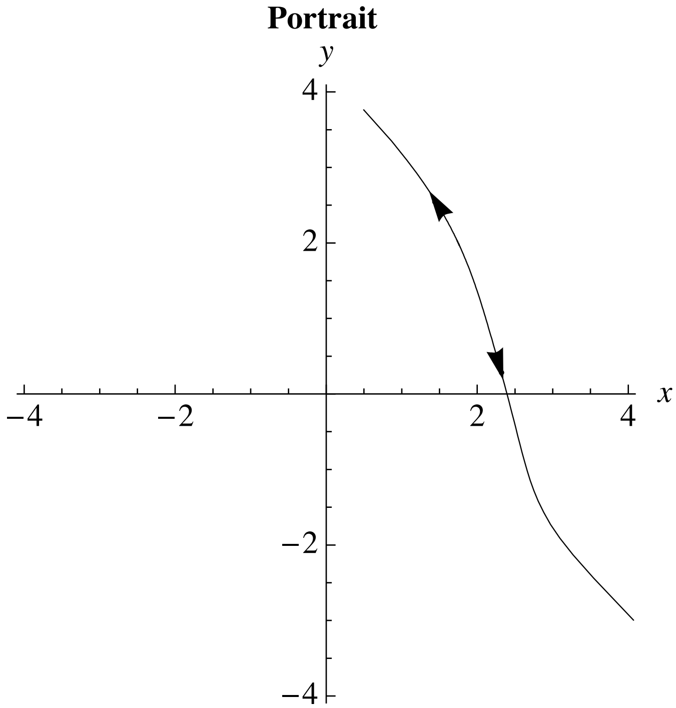
```
eigvec12 = Last[Last[eig1]]
```

```
{-0.793775, -0.608212}
```

Notice how different the eigenvalues are in magnitude. Motion out along the unstable manifold will be much more rapid than motion in along the stable manifold. We construct the unstable manifold first.

```
ε = 0.005;

initset = {eq1 + ε * eigvec11, eq1 - ε * eigvec11};

t0 = 0.0; h = 0.002; nsteps = 5000;

arrowflag = True; arrowvec = {1 / 3};
```

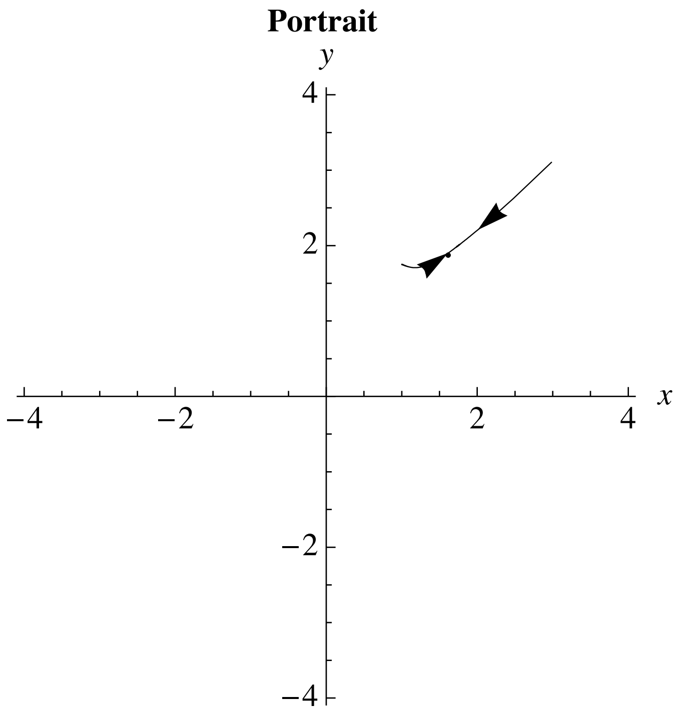
```
graph11 = portrait[initset, t0, h, nsteps, 1, 2]
```



Now the stable manifold.

```
initset = {eq1 + ε * eigvec12, eq1 - ε * eigvec12};

t0 = 0.0; h = -0.02; nsteps = 5000;

arrowvec = {4 / 5};
```

```
graph12 = portrait[initset, t0, h, nsteps, 1, 2]
```



Now we look at the second equilibrium, eq2.

```
eig2 = eigsys[eq2]
```

```
{{29.2593, -11.0901}, {{0.347401, -0.937717}, {-0.927012, -0.375032}}}
```

```
eigvec21 = First[Last[eig2]]
```

```
{0.347401, -0.937717}
```

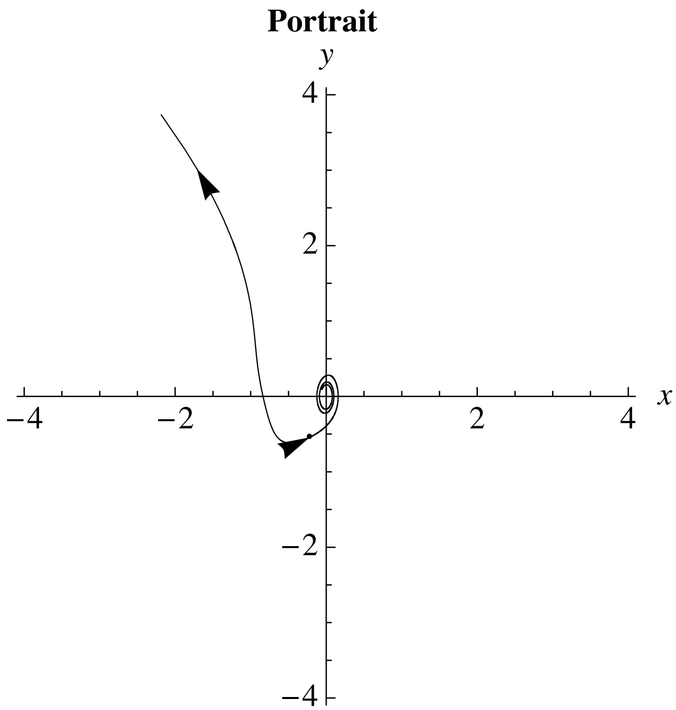
```
eigvec22 = Last[Last[eig2]]
```

```
{-0.927012, -0.375032}
```

First the unstable manifold.

```
initset = {eq2 + ε * eigvec21, eq2 - ε * eigvec21};
t0 = 0.0; h = 0.002; nsteps = 5000;
arrowflag = True; arrowvec = {1 / 2};
```

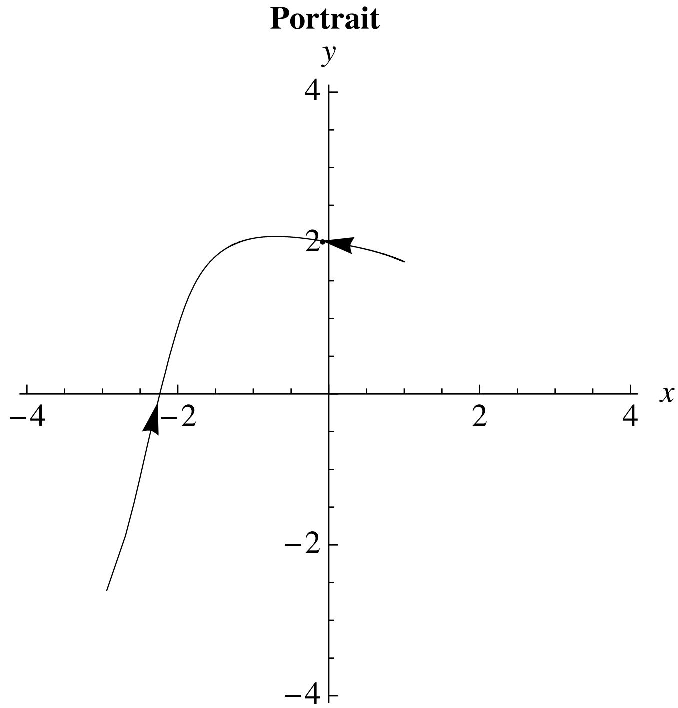
```
graph21 = portrait[initset, t0, h, nsteps, 1, 2]
```



Now the stable manifold.

```
initset = {eq2 + ε * eigvec22, eq2 - ε * eigvec22};
t0 = 0.0; h = -0.002; nsteps = 5000;
```

```
graph22 = portrait[initset, t0, h, nsteps, 1, 2]
```



Next we look at the last saddle, eq4.

```
eig4 = eigsys[eq4]
```

{{-6.1224, 1.21753}, {{-0.232615, 0.972569}, {0.910893, 0.412642}}}

```
eigvec41 = First[Last[eig4]]
```

{-0.232615, 0.972569}

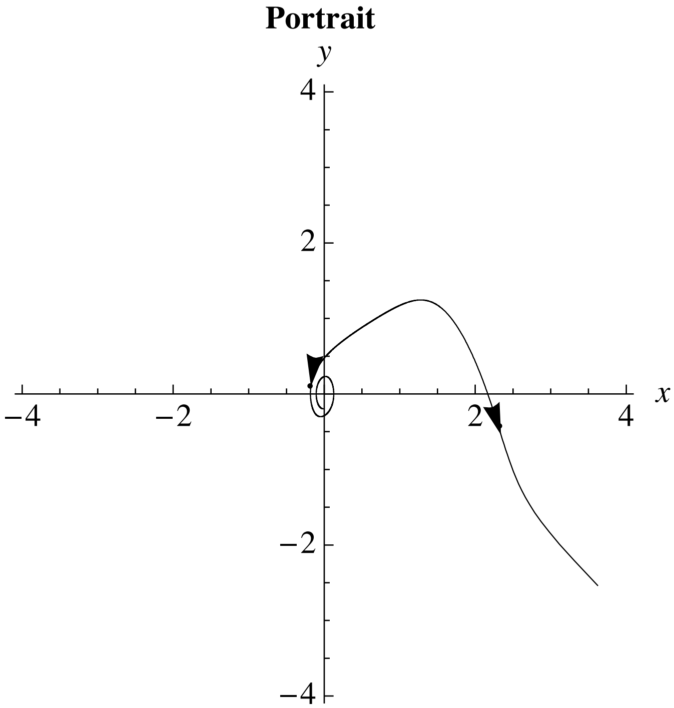
```
eigvec42 = Last[Last[eig4]]
```

{0.910893, 0.412642}

First the unstable manifold.

```
initset = {eq4 + ε * eigvec42, eq4 - ε * eigvec42};
t0 = 0.0; h = 0.002; nsteps = 5000;
arrowflag = True; arrowvec = {1 / 2};
```

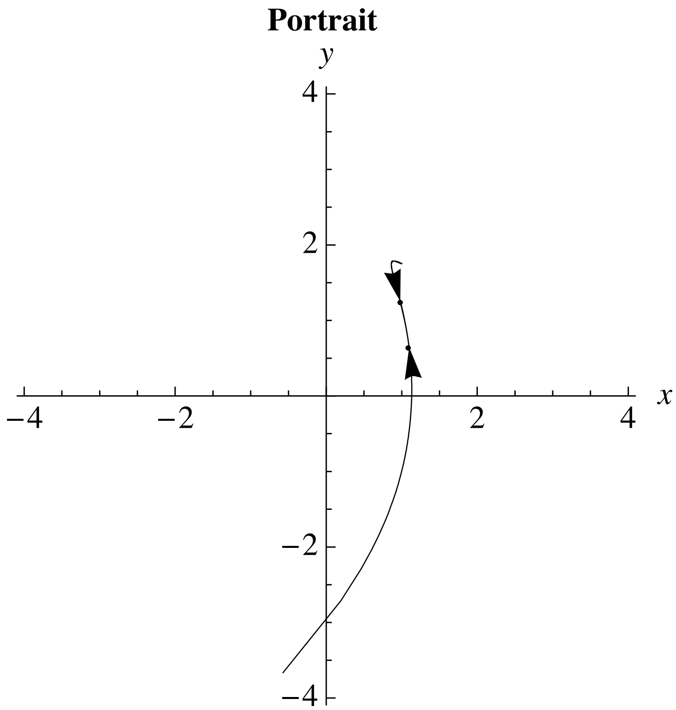
```
graph42 = portrait[initset, t0, h, nsteps, 1, 2]
```



Now the stable manifold.

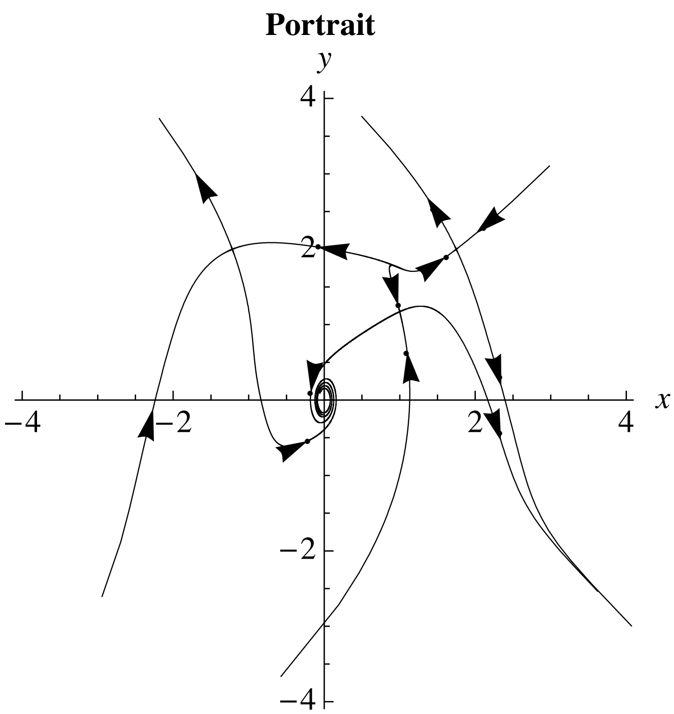
```
bothdirflag = False;

initset = {eq4 + ε * eigvec41, eq4 - ε * eigvec41};

t0 = 0.0; h = -0.002; nsteps = {5000, 5000};

arrowvec = {9 / 10};
```

```
graph41 = portrait[initset, t0, h, nsteps, 1, 2]
```



Before looking at the unstable node eq3 and the stable spiral at the origin, we first combine the graphs we have so far.

```
firstbigraph =
 show[graph11, graph12, graph21, graph22, graph41, graph42]
```



Now we look at the unstable node eq3. Some of the integral curves leaving this point have already been constructed. A little experimenting shows that one more curve is useful.

```
eig3 = eigsys[eq3]
```

```
{{7.55853, 2.22907}, {{0.499355, -0.866398}, {-0.920325, 0.391154}}}
```

```
eigvec31 = First[Last[eig3]]
```

```
{0.499355, -0.866398}
```

```
eigvec32 = Last[Last[eig3]]
```

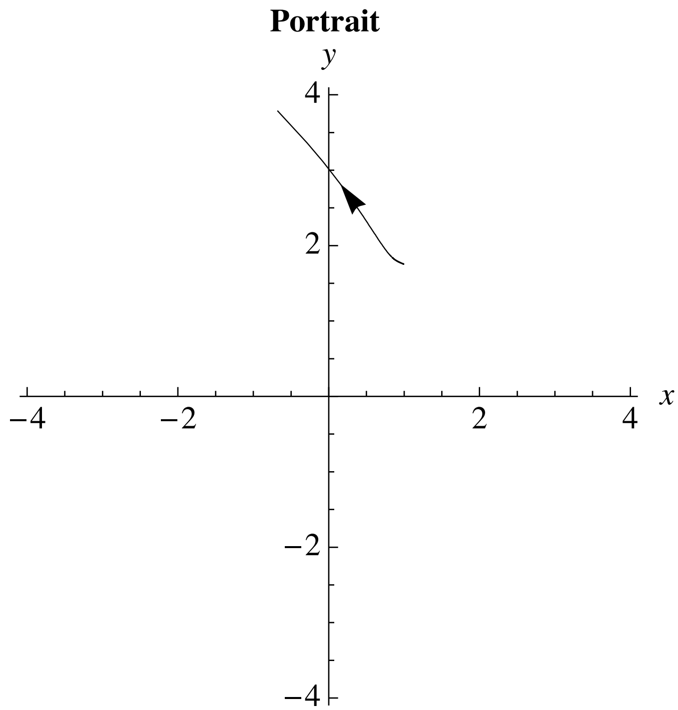
```
{-0.920325, 0.391154}
```

```
initset = {eq3 + ε * eigvec32};
```

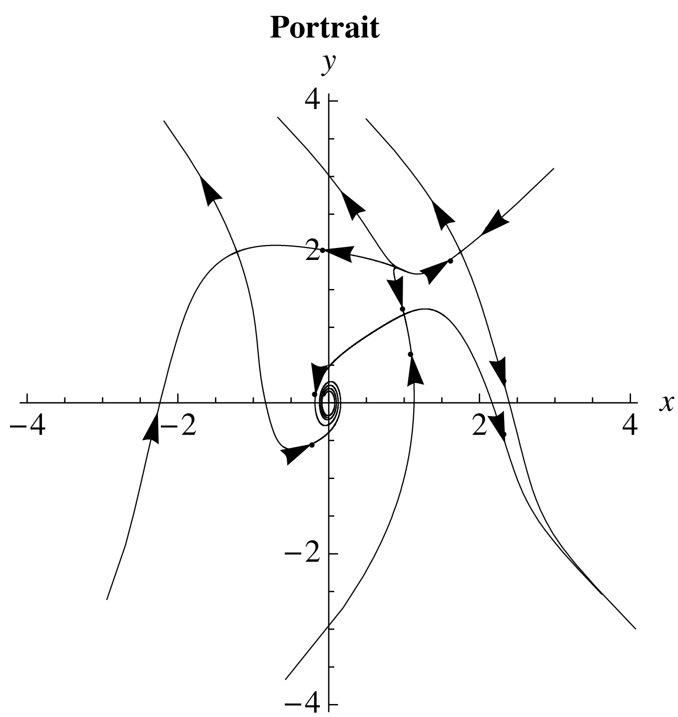
```
t0 = 0.0; h = 0.002; nsteps = 5000;
```

```
arrowvec = {1 / 2};
```

```
graph32 = portrait[initset, t0, h, nsteps, 1, 2]
```

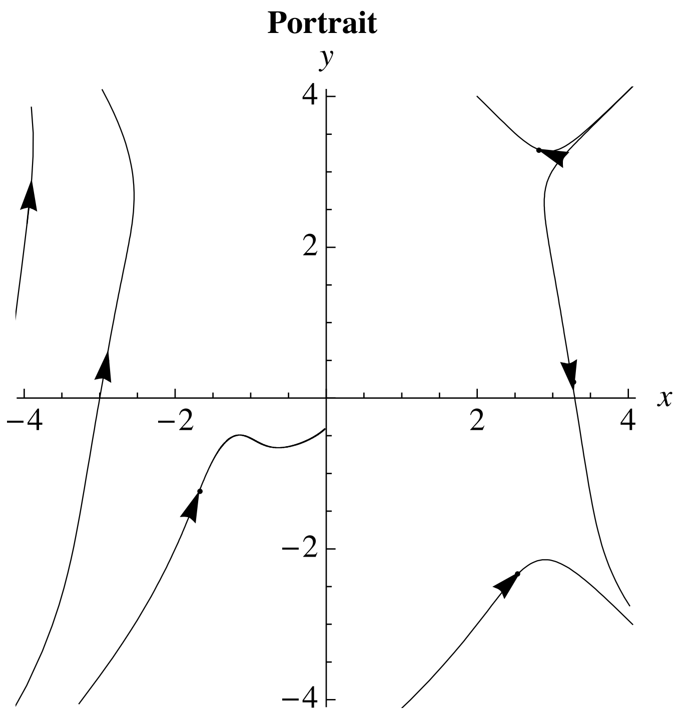



```
bigraph = show[firstbigraph, graph32]
```



We add a few more orbits to fill in the picture.

```
initset = {{3, 1.75}, {2, 4}, {2, -3}, {-3, 0}, {-4, 2}, {-2, -2}};

t0 = 0.0; h = 0.00025; nsteps = 5000;

bothdirflag = True;

arrowvec = {3 / 5};

lastpiece = portrait[initset, t0, h, nsteps, 1, 2]
```



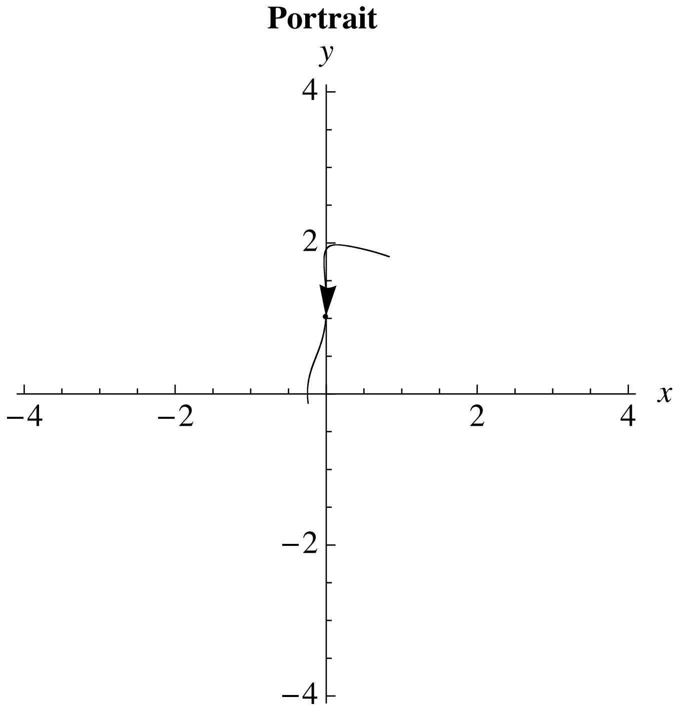
Now we put the pictures together.

```
newbigraph = show[bigraph, lastpiece];
```

One more orbit.

```
bothdirflag = True; init = {0, 1.25}; h = 0.00025; nsteps = 4000; t0 = 0.0;

solast = integrate[init, t0, h, nsteps];
```

```
lastpiece = phaser[solast]
```




```
newbigraph2 = show[newbigraph, lastpiece];
```

As a final touch, we place a blue dot on the stable equilibrium and red dots on the saddles, and a green dot on the unstable node.

```
setcolor[{Red, Blue, Green}];

ptsize = 0.03;

dotgraph1 = dots[{eq1, eq2, eq4}];

dotgraph2 = dots[{eq5}];

dotgraph3 = dots[{eq3}];

imsize = 450;
```

```
show[dotgraph1, dotgraph2, dotgraph3, newbigraph2]
```