ME 406 A Catalogue of Equilibria for Systems of Two Linear Autonomous First Order Equations

■ Introduction

In this notebook, we look at solutions of equations of the following form:

$$\frac{dx}{dt} = a\,x + b\,y\ ,$$

$$\frac{dy}{dt} = c\,x + d\,y\ .$$

Here a, b, c and d are constants, and, unless otherwise specified, we assume that

$$a\,d - b\,c \neq 0.$$

These equations have solutions of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} e^{r t}\ ,$$

where r, u, and v satisfy

$$(a - r)\,u + b\,v = 0\ ,$$

$$c\,u + (d - r)\,v = 0.$$

For these two homogeneous linear equations to have a non-trivial solution, the determinant of the coefficient array must vanish. Thus

$$r^2 - (a + d)\,r + (a\,d - b\,c) = 0\ .$$

Each of the two roots of this quadratic determines a solution. The solution is completed by solving the equations for u and v for each value of r. Because the equations are homogeneous, the vector (u,v) is determined only to within a constant