ME 406
Bifurcations VII
Subcritical Hopf Bifurcation

```
sysid
```

```
Mathematica  6.0.3, DynPac 11.01, 1/12/2009
```

```
intreset; plotreset; imsize = 250;
```

---

1. Introduction

In this notebook, the seventh in a series of notebooks on bifurcations, we look at a simple example of a subcritical Hopf bifurcation. At the bifurcation, an unstable limit cycle is absorbed by a stable spiral equilibrium. The equilibrium becomes unstable, and the system state will then jump from the unstable equilibrium to some distant atractor, a possibly catastrophic transition. We construct a movie showing the changes in the equilibrium and the limit cycles as the bifurcation parameter changes.

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2. Definition of the System

We consider the following system, depending on one parameter *a*:

$$\dot{x} = -y + x(x^2 + y^2 + \mu)(1 - x^2 - y^2) \;,\; \dot{y} = x + y(x^2 + y^2 + \mu)(1 - x^2 - y^2)$$

We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}];
slopevec =
  {-y + x * (x^2 + y^2  + μ) * (1 - x^2 - y^2), x + y * (x^2 + y^2  + μ) * (1 - x^2 - y^2)};
sysname = "SubCritical Hopf";
```

We look for equilibrium states.

```
findpolyeq
```

```
{{0, 0}}
```

It is easy to show that this is the only equilibrium. For this problem, it is useful to obtain the slope functions in polar coordinates. We use the function polartrans to do this.

```
polarslope = polartrans[x, y, r, θ]
```

```
{-r (-1 + r^2) (r^2 + μ), 1}
```

We have gotten lucky. Almost everything we might want to know about this system can be read from the radial slope function. We see that there is a limit cycle at $r = 1$, and, for negative μ, another at $r = \sqrt{-\mu}$. It is easy to check the stability, by looking at the sign of polarslope as we move slightly away from the limit cycle. We get the following results: for $\mu < -1$, the cycle at $\sqrt{-\mu}$ is stable, and the cycle at 1 is unstable. For $-1 < \mu < 0$, the cycle at $\sqrt{-\mu}$ is unstable, and the cycle at 1 is stable. Thus we see that as μ passes through -1, we have a transcritical bifurcation of limit cycles. As $\mu \to 0-$, the unstable cycle is absorbed by the equilibrium. For $0 < \mu$, we have only the stable cycle at 1. Let's check the stability of the equilibrium point.

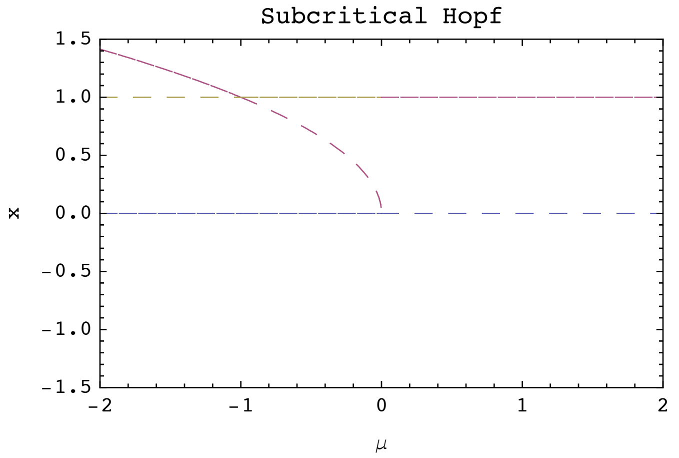
```
eigsys[{0, 0}]
```

```
{{-ⅈ + μ, ⅈ + μ}, {{-ⅈ, 1}, {ⅈ, 1}}}
```

Thus the equilibrium is a stable spiral for $\mu < 0$, an unstable spiral for $\mu > 0$, and a center in linearized theory for $\mu = 0$. It is not hard to show that the equilibrium is unstable for $\mu = 0$. The bifurcation at $\mu = 0$ is called a subcritical Hopf bifurcation. Imagine a system in the stable equilibrium state for $\mu < 0$. As we increase μ, the unstable limit cycle shrinks to the equilibrium. When μ reaches zero, the equilibrium becomes unstable, and the system will make a finite jump to the nearest attractor, namely the limit cycle at $r = 1$. Such a large and sudden jump could be catastrophic. A less catastrophic sequence occurs if we start on the stable limit cycle for $\mu < -1$. Then as we pass through $\mu = -1$, we switch to the stable cycle at $r = 1$, and remain there as μ continues to increase. We may visualize this in a bifurcation diagram by plotting the position of the equilibria and the amplitude of the limit cycles, using as usual solid for stable, dashed for unstable.

```
plot1 = Plot[{0, √-μ , 1}, {μ, -2, -1},
   PlotRange → {{-2, 2}, {-1.5, 1.5}},
   PlotLabel → "Subcritical Hopf",
   FrameLabel → {"μ", "x"}, Axes → False, ImageSize → imsize,
   Frame → True,
   PlotStyle → {Dashing[{0.03, 0.0}], Dashing[{0.03, 0.0}],
     Dashing[{0.03, 0.03}]}];

plot2 = Plot[{0, √-μ , 1}, {μ, -1, 0},
   PlotRange → {{-2, 2}, {-1.5, 1.5}},
   PlotLabel → "Subcritical Hopf", Frame → True,
   FrameLabel → {"μ", "x"}, Axes → False,
   ImageSize → imsize,
   PlotStyle → {Dashing[{0.03, 0.00}], Dashing[{0.03, 0.03}],
     Dashing[{0.03, 0.0}]}];

plot3 = Plot[{0, 1}, {μ, 0, 2}, PlotRange → {{-2, 2}, {-1.5, 1.5}},
   PlotLabel → "Subcritical Hopf", Frame → True,
   FrameLabel → {"μ", "x"}, Axes → False,
   PlotStyle → {Dashing[{0.03, 0.03}], Dashing[{0.03, 0.00}]}];
```

```
Show[{plot1, plot2, plot3}]
```

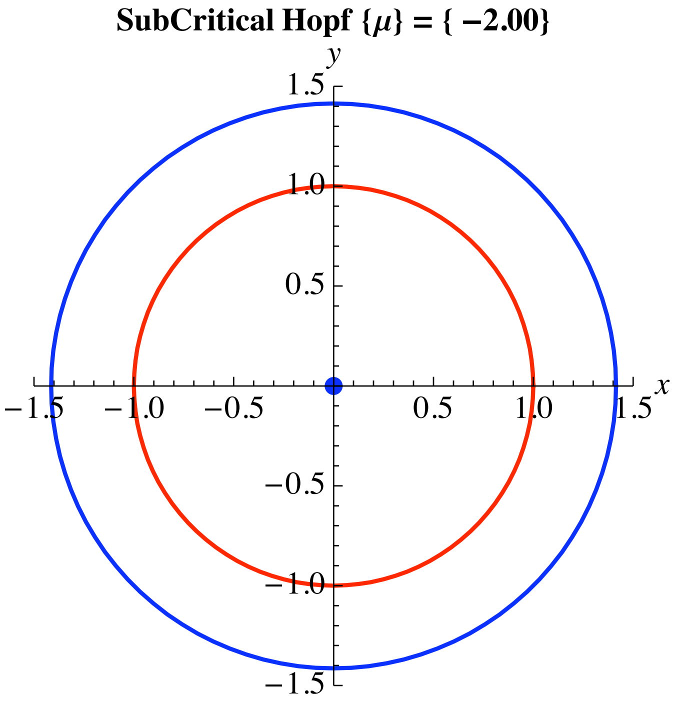


In previous bifurcation sequences, we have constructed a family of orbits for each μ, and then tracked the changes with μ. Because there is more going on here with the two limit cycles and the equilibrium, we first make a movie showing only the limit cycles and equilibrium. This shows the essential features of the bifurcation. Then we will make a second movie in which we add two orbits near the equilibrium point. The refgraph we define here has the limit cycles and equilibrium as a function of mu.

```
refgraph1 := Module[{ans1, ans2, ans3, rad, u},
  ptsize = 0.03; rad = Sqrt[-First[parmval]];
  setcolor[{Blue}]; ans1 = dots[{{0, 0}}]; lnthick = 0.007;
  ans2 = plotcurve[{{{rad * Cos[u], rad * Sin[u]}, {u, 0, 2 Pi}}}];
  setcolor[{Red}];
  ans3 = plotcurve[{{{Cos[u], Sin[u]}, {u, 0, 2 Pi}}}];
  lnthick = 0.002; {ans1, ans2, ans3}]

refgraph2 := Module[{ans1, ans2, ans3, rad, u},
  ptsize = 0.03; rad = Sqrt[-First[parmval]];
  setcolor[{Blue}]; ans1 = dots[{{0, 0}}]; lnthick = 0.007;
  ans2 = plotcurve[{{{Cos[u], Sin[u]}, {u, 0, 2 Pi}}}];
  setcolor[{Red}];
  ans3 = plotcurve[{{{rad * Cos[u], rad * Sin[u]}, {u, 0, 2 Pi}}}];
  lnthick = 0.002; {ans1, ans2, ans3}]

refgraph3 := Module[{ans1, ans2, u}, ptsize = 0.03;
  setcolor[{Red}]; ans1 = dots[{{0, 0}}];
  setcolor[{Blue}]; lnthick = 0.007;
  ans2 = plotcurve[{{{Cos[u], Sin[u]}, {u, 0, 2 Pi}}}];
  lnthick = 0.002; {ans1, ans2}]

refgraph := Module[{ans, mu}, mu = First[parmval];
  plrange = {{-1.5, 1.5}, {-1.5, 1.5}}; asprat = 1;
  If[(mu < -1), (ans = refgraph1), (If[(mu < 0), (ans = refgraph2),
     (ans = refgraph3) ])];
  setcolor[{Black}]; display = True; ans]
```

Now we use a Do loop to construct the movie with just the limit cycles and the equilibrium. As usual we use blue for stable and red for unstable.

```
Do[parmval = {i * 0.05}; Print[show[refgraph]], {i, -40, 20}]
```



Now we repeat the graph construction, this time adding two orbits. Because varying number of steps are required for different values of μ, we have defined nsteps below to accomplish this.

```
t0 = 0.0; h = 0.02; initset = { {0.1, 0}, {-0.1, 0}};
```

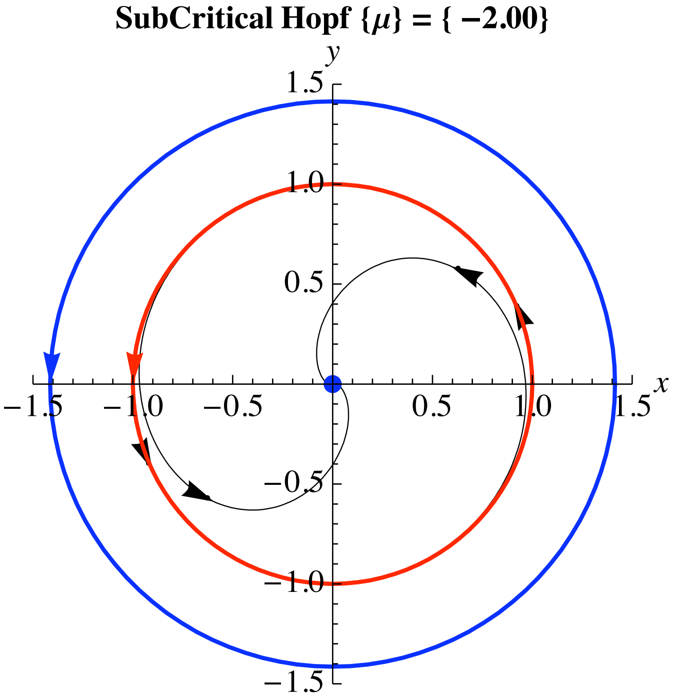
```
bothdirflag = True;
```

```
arrowflag = True; arrowvec = {1 / 5, 4 / 5};
```

```
parmlist = Module[{ans, i}, ans = {};
   Do[ans = Append[ans, {i * 0.05}], {i, -40, 20}]; ans];
```

```
nsteps := Module[{ans, mu}, mu = First[parmval];
  If[(mu < -1.45), (ans = 350), (If[(mu < -0.95), (ans = 600),
      (If[(mu < -0.25), (ans = 700),
        (If[(mu < -0.2), (ans = 800),
          (If[(mu < -0.15), (ans = 900),
            (If[(mu < -0.10), (ans = 1000),
              (If[(mu < -0.05), (ans = 1200),
                (If[(mu == 0 ), (ans = 5000),
                  (If[(mu < 0.1), (ans = 1400),
                    (If[(mu < 0.2), (ans = 1200),
                      (If[(mu < 0.25), (ans = 1000),
                        (If[(mu < 0.3), (ans = 900),
                          (If[(mu < 0.35), (ans = 800),
                            (If[(mu < 0.4), (ans = 700),
                              (If[(mu < 0.45), (ans = 600),
                                (ans =
                                  500)])])])])])])])])])])])])])])]]

initset := Module[{ans, mu}, mu = First[parmval];
  If[((mu < -0.25) || (mu > 0.25)), (ans = {{0.02, 0}, {-0.02, 0}}),
   (ans = {{0.1, 0}, {-0.1, 0}})]; ans]

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]
```

```
Bifurcation sequence for parmlist =
 {{-2.}, {-1.95}, {-1.9}, {-1.85}, {-1.8}, {-1.75}, {-1.7}, {-1.65},
  {-1.6}, {-1.55}, {-1.5}, {-1.45}, {-1.4}, {-1.35}, {-1.3}, {-1.25},
  {-1.2}, {-1.15}, {-1.1}, {-1.05}, {-1.}, {-0.95}, {-0.9}, {-0.85},
  {-0.8}, {-0.75}, {-0.7}, {-0.65}, {-0.6}, {-0.55}, {-0.5}, {-0.45},
  {-0.4}, {-0.35}, {-0.3}, {-0.25}, {-0.2}, {-0.15}, {-0.1}, {-0.05}, {0},
  {0.05}, {0.1}, {0.15}, {0.2}, {0.25}, {0.3}, {0.35}, {0.4}, {0.45}, {0.5},
  {0.55}, {0.6}, {0.65}, {0.7}, {0.75}, {0.8}, {0.85}, {0.9}, {0.95}, {1.}}
```


SubCritical Hopf {μ} = { −2.00}
y
1.5
1.0
0.5
−0.5
−1.0
−1.5
x
−1.5
−1.0
−0.5
0.5
1.0
1.5