

# ME 406

## Bifurcations VI

### Supercritical Hopf Bifurcation

```
sysid
```

```
Mathematica 6.0.3, DynPac 11.01, 1/12/2009
```

```
intreset; plotreset; imsize = 250;
```

## 1. Introduction

In this notebook, the sixth in a series of notebooks on bifurcations, we look at a simple example of a supercritical Hopf bifurcation. At the bifurcation, a stable spiral equilibrium becomes an unstable spiral equilibrium and throws off a stable limit cycle. The amplitude of the limit cycle at birth is zero, and increases as the bifurcation parameter increases beyond the bifurcation value. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

## 2. Definition of the System

We consider the following system, depending on one parameter  $\mu$ :

$$\dot{x} = y - x(x^2 + y^2 - \mu), \quad \dot{y} = -x - y(x^2 + y^2 - \mu).$$

We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}];  
slopevec = {y - x * (x^2 + y^2 - μ), -x - y * (x^2 + y^2 - μ)};  
sysname = "SuperCritical Hopf";
```

We look for equilibrium states.

```
findpolyeq
```

```
{{0, 0}}
```

It is easy to show that this is the only equilibrium. For this problem, it is useful to obtain the equation satisfied by the polar radial coordinate  $r = (x^2 + y^2)^{\frac{1}{2}}$ . We have

```
r = FullSimplify[((1/r) (x * slopevec[[1]] + y * slopevec[[2]])) //.  
  x^2 -> (r^2 - y^2)  
  -r (r^2 - μ)
```

This equation shows clearly that there is a stable limit cycle for  $\mu > 0$ , namely a circular limit cycle of radius  $\sqrt{\mu}$ . For  $\mu < 0$ ,  $\dot{r}$  is negative for all  $r$  except  $r = 0$ , which shows that the origin is a globally stable equilibrium in that case. We look at the equilibrium in more detail.

```
eigsys[{0, 0}]
{{-i + μ, i + μ}, {{i, 1}, {-i, 1}}}
```

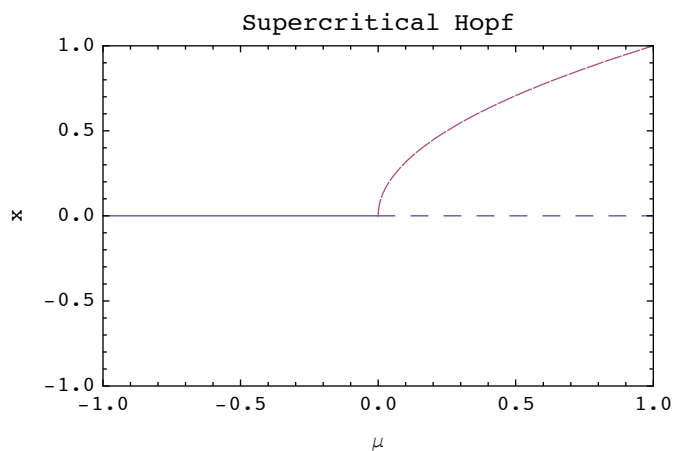
Thus the equilibrium is a stable spiral for  $\mu < 0$ , an unstable spiral for  $\mu > 0$ , and a center in linearized theory for  $\mu = 0$ . It is not hard to show that the equilibrium is a stable nonlinear spiral for  $\mu = 0$ . The bifurcation at  $\mu = 0$  is called a Hopf bifurcation. We see that the equilibrium goes from a stable to an unstable spiral there, and a limit cycle of zero amplitude is born there. As the parameter increases beyond the bifurcation value (zero in this case), the limit cycle grows in amplitude.

We may visualize this in a bifurcation diagram by plotting the position of the equilibria and the amplitude of any limit cycles.

```
plot1 = Plot[{0}, {μ, -1, 0},
  PlotRange → {{-1, 1}, {-1, 1}},
  PlotLabel → "Supercritical Hopf",
  FrameLabel → {"μ", "x"},
  Axes → False, ImageSize → imsize, Frame → True];

plot2 = Plot[{0, √μ}, {μ, 0, 1},
  PlotRange → {{-1, 1}, {-1, 1}},
  PlotLabel → "Supercritical Hopf", Frame → True,
  FrameLabel → {"μ", "x"}, Axes → False,
  ImageSize → imsize,
  PlotStyle → {Dashing[{0.03, 0.03}], Dashing[{0.03, 0.0}]}];

Show[{plot1, plot2}]
```



Imagine a gradual change of  $\mu$ -values from negative through zero to small and positive. For negative  $\mu$ , the system will settle into the only attractor, the stable spiral at  $x = 0$ . It will stay there as we increase  $\mu$  slowly.

When we just exceed  $\mu = 0$ , the equilibrium is no longer stable, and the system will make the small jump to the limit cycle of amplitude  $\sqrt{\mu}$ .

Now we construct a short sequence of phase plots for different values of  $\mu$ , for a given set of initial conditions. These will illustrate the bifurcation at  $\mu = 0$ . We will mark the equilibria by red dots for unstable, blue dots for stable. We do this by constructing a graph `refgraph` which is then included in each picture of the bifurcation sequence.

```

refgraph := Module[{ans}, psize = 0.025;
  display = False; setcolor[{Black}];
  If[(First[parmval] <= 0), (setcolor[{Blue}]);
    ans = dots[{{0, 0}}], (setcolor[{Red}]);
    ans = dots[{{0, 0}}]]];
  setcolor[{Black}]; display = True; ans]

ε = 0.02;

initset1 =
  {{2, 0}, {2, 2}, {0, 2}, {-2, 2}, {-2, 0}, {-2, -2}, {0, -2}, {2, -2}};

initset2 = {{2, 0}, {2, 2}, {0, 2}, {-2, 2},
  {-2, 0}, {-2, -2}, {0, -2}, {2, -2}, {ε, 0}, {-ε, 0}};

initset := Module[{ans},
  If[(First[parmval] <= 0), (ans = initset1), (ans = initset2)]; ans]

plrange = {{-2, 2}, {-2, 2}}; asprat = 1;

arrowflag = True; arrowvec = {1 / 5};

t0 = 0.0; h = 0.02; nsteps = 500; bothdirflag = False;

rangeflag = False;

```

Now we choose a small number of  $\mu$ -values for a bifurcation sequence.

```

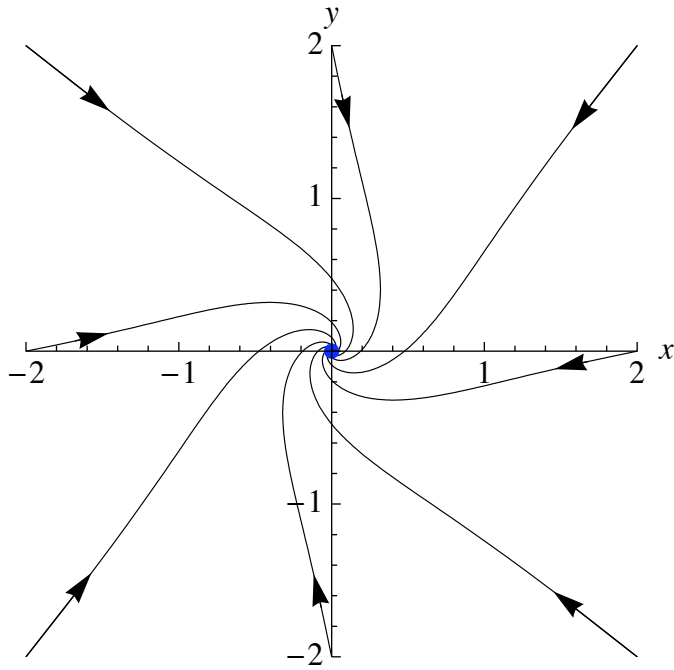
parmlist = {{-1.0}, {-0.5}, {0.0}, {0.5}, {1.0}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]

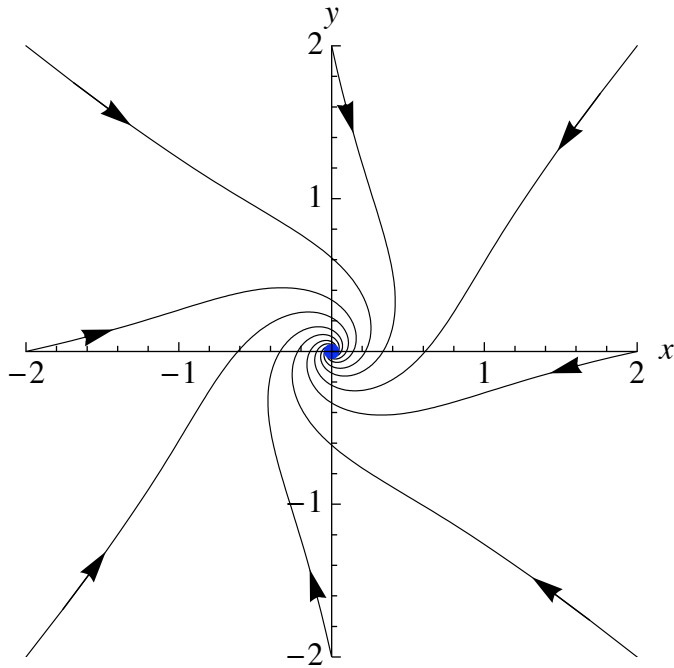
Bifurcation sequence for parmlist = {{-1.}, {-0.5}, {0.}, {0.5}, {1.}}

```

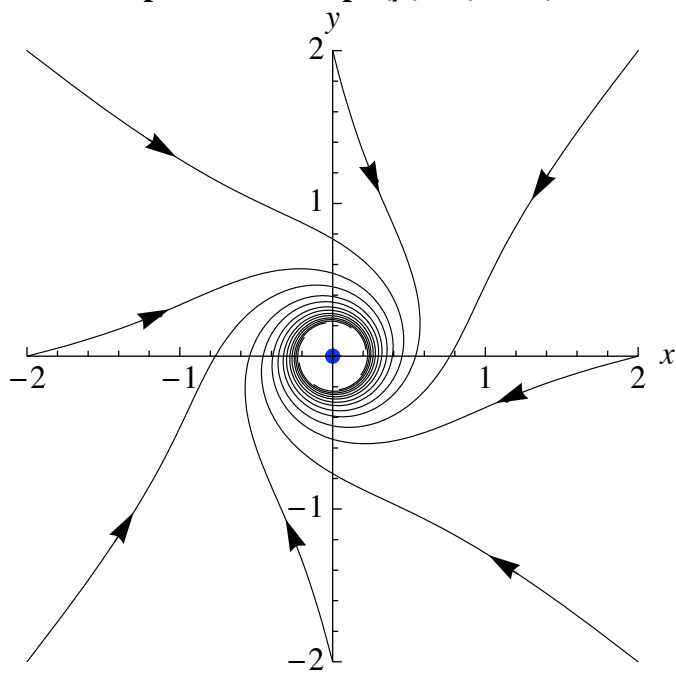
**SuperCritical Hopf  $\{\mu\} = \{-1.00\}$**



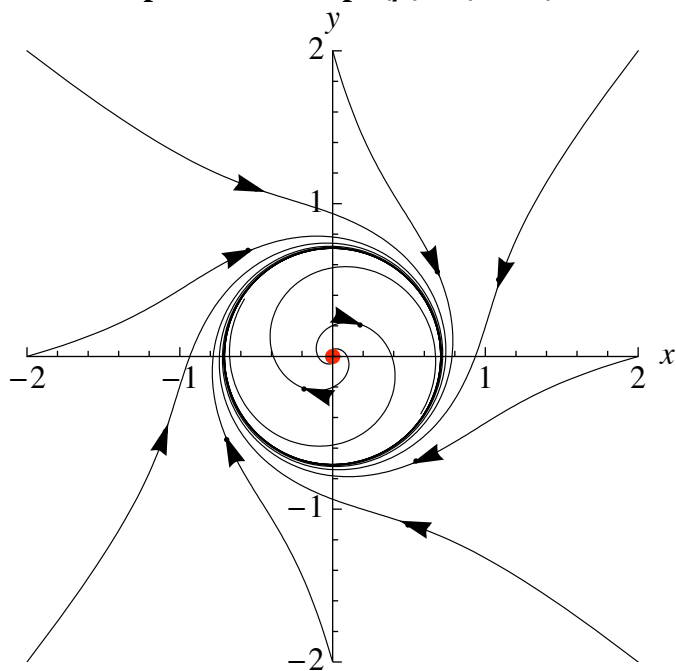
**SuperCritical Hopf  $\{\mu\} = \{-0.50\}$**



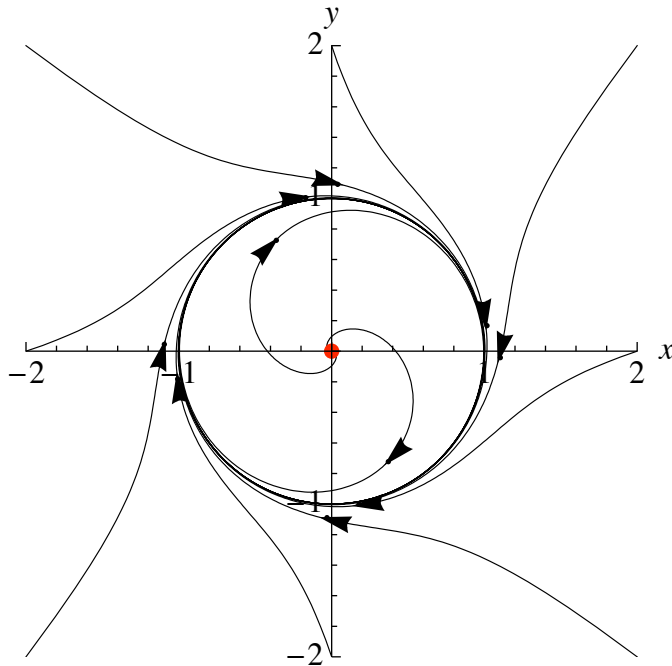
**SuperCritical Hopf  $\{\mu\} = \{ 0.00\}$**



**SuperCritical Hopf  $\{\mu\} = \{ 0.50\}$**



### SuperCritical Hopf $\{\mu\} = \{ 1.00\}$



As our final task in this notebook, we will make a movie of the subcritical pitchfork bifurcation. We will center the frames of the movie around  $\mu = 0$ . We make 41 frames at  $\mu$ -intervals of 0.05, with  $\mu$  running from -1 to 1. We use the colored dots for equilibria as before. We use a parameter-dependent time step so that we can use more time steps to capture the nonlinear spiral near  $\mu = 0$ .

```

nsteps := Module[{mew, ans},
  mew = First[parmval]; If[(mew < -0.25), (ans = 500),
    (If[(mew < 0), (ans = 2000),
      (If[(mew == 0), (ans = 5000),
        (If[(mew < 0.1),
          (ans = {500, 500, 500, 500, 500, 500, 500, 500, 5000, 5000})],
          (If[(mew < 0.5),
            (ans = {500, 500, 500, 500, 500, 500, 500, 500, 2000, 2000})],
            (ans = 500)])))]]; ans]

parmlist = Module[{ans, i}, ans = {};
  Do[ans = Append[ans, {0.05 * i}], {i, -20, 20}]; ans]

{{-1.}, {-0.95}, {-0.9}, {-0.85}, {-0.8}, {-0.75}, {-0.7}, {-0.65},
{-0.6}, {-0.55}, {-0.5}, {-0.45}, {-0.4}, {-0.35}, {-0.3},
{-0.25}, {-0.2}, {-0.15}, {-0.1}, {-0.05}, {0}, {0.05}, {0.1},
{0.15}, {0.2}, {0.25}, {0.3}, {0.35}, {0.4}, {0.45}, {0.5}, {0.55},
{0.6}, {0.65}, {0.7}, {0.75}, {0.8}, {0.85}, {0.9}, {0.95}, {1.}}

```

```
bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]
```

Bifurcation sequence for parmlist =

```
{{-1.}, {-0.95}, {-0.9}, {-0.85}, {-0.8}, {-0.75}, {-0.7},  
{-0.65}, {-0.6}, {-0.55}, {-0.5}, {-0.45}, {-0.4}, {-0.35}, {-0.3},  
{-0.25}, {-0.2}, {-0.15}, {-0.1}, {-0.05}, {0}, {0.05}, {0.1},  
{0.15}, {0.2}, {0.25}, {0.3}, {0.35}, {0.4}, {0.45}, {0.5}, {0.55},  
{0.6}, {0.65}, {0.7}, {0.75}, {0.8}, {0.85}, {0.9}, {0.95}, {1.}}
```

**SuperCritical Hopf  $\{\mu\} = \{-1.00\}$**

