ME 406
Bifurcations V
Subcritical Pitchfork Bifurcation

```
sysid
```

```
Mathematica  6.0.3, DynPac 11.01, 1/12/2009
```

```
intreset; plotreset; imsize = 250;
```

1. Introduction

In this notebook, the fifth in a series of notebooks on bifurcations, we look at a simple example of a subrcritical pitchfork bifurcation. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

2. Definition of the System

We consider the following system, depending on one parameter *a*:

$$\dot{x} = \mu x + x^3 \ , \ \ \dot{y} = -y \ .$$

This system has one equilibrium for any $\mu \geq 0$, and three equilibria for $\mu < 0$. The bifurcation is $\mu = 0$, for which three equilbria coalesce into one as we increase μ. We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}]; slopevec = {μ x + x^3, - y};
sysname = "Subcritical Pitchfork";
```

```
eq1 = {0, 0}; eq2 = {Sqrt[-μ], 0}; eq3 = {-Sqrt[-μ], 0};
```

Let's look at the nature of the equilibria.

```
eigsys[eq1]
```

```
{{-1, μ}, {{0, 1}, {1, 0}}}
```

```
eigsys[eq2]
```

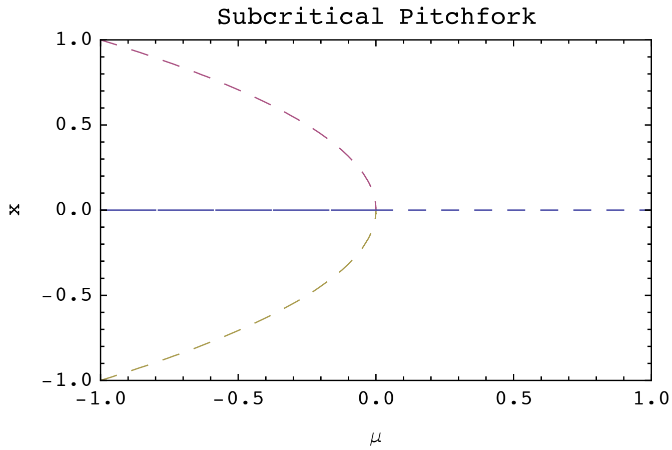
```
{{-1, -2 μ}, {{0, 1}, {1, 0}}}
```

```
eigsys[eq3]
```

```
{{-1, -2 μ}, {{0, 1}, {1, 0}}}
```

Thus for $\mu < 0$, eq1 is a stable node, and both eq2 and eq3 are saddles and therefore unstable. For $\mu > 0$, the only equilibrium eq1 is a saddle. For $\mu = 0$, linearization is inconclusive, although in this case it is easy to integrate the equations directly and show that the equilibrium is unstable. This kind of bifurcation is called a subcritical pitchfork. We may visualize this with a bifurcation diagram, showing the equilibria as functions of μ, with stable in solid, unstable in dashed.

```
plot1 = Plot[{0, √(-μ), -√(-μ)}, {μ, -1, 0},
   PlotRange → {{-1, 1}, {-1, 1}},
   PlotLabel → "Subcritical Pitchfork",
   FrameLabel → {"μ", "x"}, Axes → False,
   ImageSize → imsize, PlotStyle → {Dashing[{0.1, 0}],
     Dashing[{0.03, 0.03}], Dashing[{0.03, 0.03}]}, Frame → True];

plot2 = Plot[{0}, {μ, 0, 1},
   PlotRange → {{-1, 1}, {-1, 1}},
   PlotLabel → "Subcritical Pitchfork", Frame → True,
   FrameLabel → {"μ", "x"}, Axes → False,
   DisplayFunction → Identity,
   ImageSize → imsize, PlotStyle → {Dashing[{0.03, 0.03}]}];

Show[{plot1, plot2}]
```



Imagine a gradual change of μ-values from negative through zero to small and positive. For negative μ, the system will settle into the only attractor, the stable node at $x = 0$. It will stay here as we increase μ slowly. When we just exceed $\mu = 0$, there are no longer any stable, and the system will make a large jump, either to ∞, or if the equations are a local approximation to some more complicated equations, to a distant attractor. Such large jumps can be catastrophic.

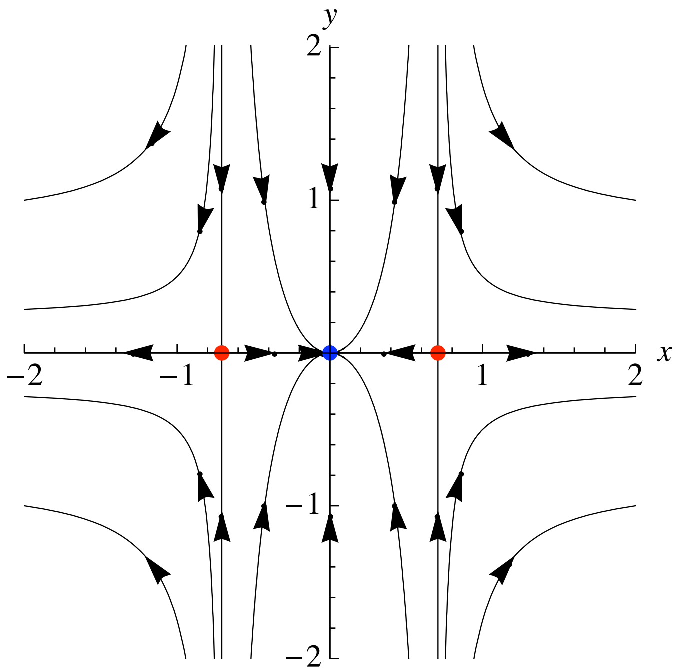
Now we construct a short sequence of phase plots for different values of μ, for a given set of initial conditions. These will illustrate the bifurcation at $\mu = 0$. We will mark the equilibria by red dots for unstable, blue dots for stable. We do this by constructing a graph refgraph which is then included in each picture of the bifurcation sequence.

```
refgraph := Module[{temp1, temp2, temp3, temp4, ans}, ptsize = 0.025;
  display = False; setcolor[{Black}];
  temp4 = plotcurve[{{{u, 0}, {u, -1, 1}}}];
  If[(First[parmval] < 0), (setcolor[{Blue}]; temp1 = dots[{eq1}];
    setcolor[{Red}]; temp2 = dots[{eq2}]; temp3 = dots[{eq3}];
    ans = {temp4, temp1, temp2, temp3}), (setcolor[{Red}];
    temp1 = dots[{eq1}]; ans = {temp4, temp1})];
  setcolor[{Black}]; display = True; ans]

ε = 0.02;

initset1 = {{2, 0}, {2, 1}, {0, 2}, {-2, 1}, {-2, 0}, {-2, -1},
   {0, -2}, {2, -1}, {ε, 0}, {-ε, 0}, {0.25, 0.25}, {0.25, -0.25},
   {-0.25, 0.25}, {-0.25, -0.25}, {1, 0.5}, {1, -0.5}, {-1, 0.5},
   {-1, -0.5}, {√(-μ), ε}, {√(-μ), -ε}, {-√(-μ), ε}, {-√(-μ), -ε}};

initset2 = {{2, 0}, {2, 1}, {0, 2}, {-2, 1}, {-2, 0}, {-2, -1}, {0, -2},
   {2, -1}, {ε, 0}, {-ε, 0}, {0.25, 0.25}, {0.25, -0.25}, {-0.25, 0.25},
   {-0.25, -0.25}, {1, 0.5}, {1, -0.5}, {-1, 0.5}, {-1, -0.5}};

initset := Module[{ans},
  If[(First[parmval] < 0), (ans = initset1), (ans = initset2)]; ans]

plrange = {{-2, 2}, {-2, 2}}; asprat = 1;

arrowflag = True; arrowvec = {1 / 2};

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};
```

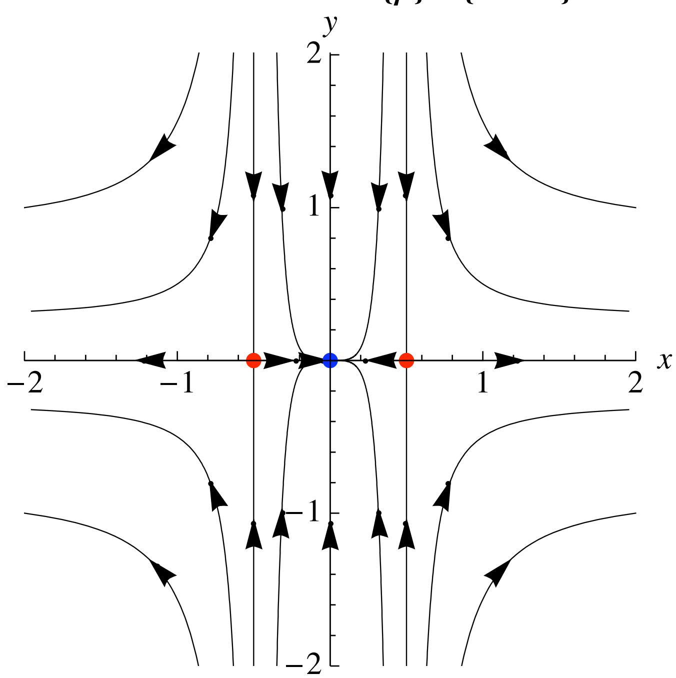
Now we choose a small number of μ-values for a bifurcation sequence.

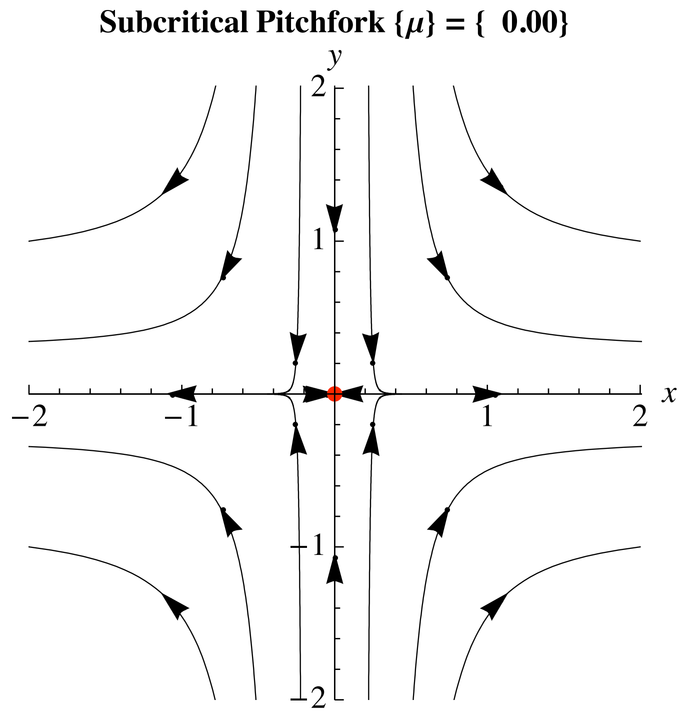
```
parmlist = {{-0.5}, {-0.25}, {0.0}, {0.25}, {0.5}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]

Bifurcation sequence for parmlist = {{-0.5}, {-0.25}, {0.}, {0.25}, {0.5}}
```

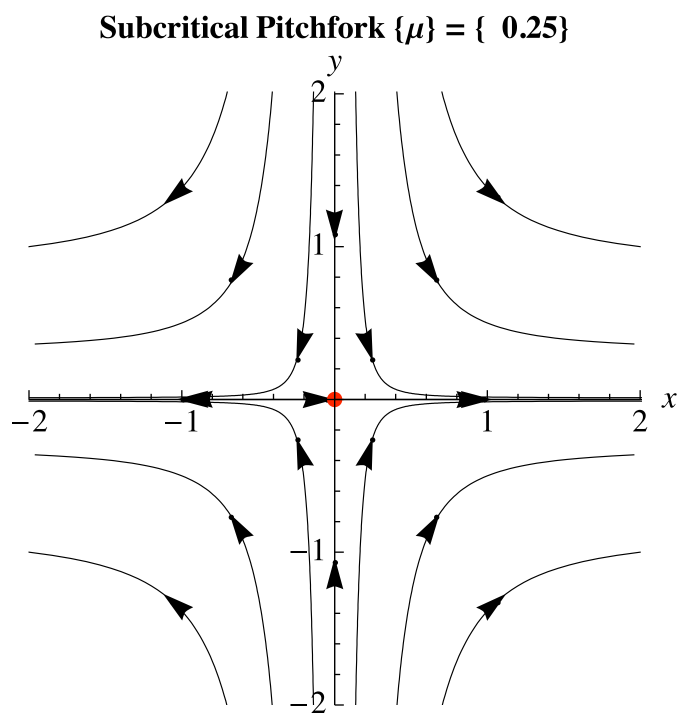
Subcritical Pitchfork $\{\mu\} = \{-0.50\}$

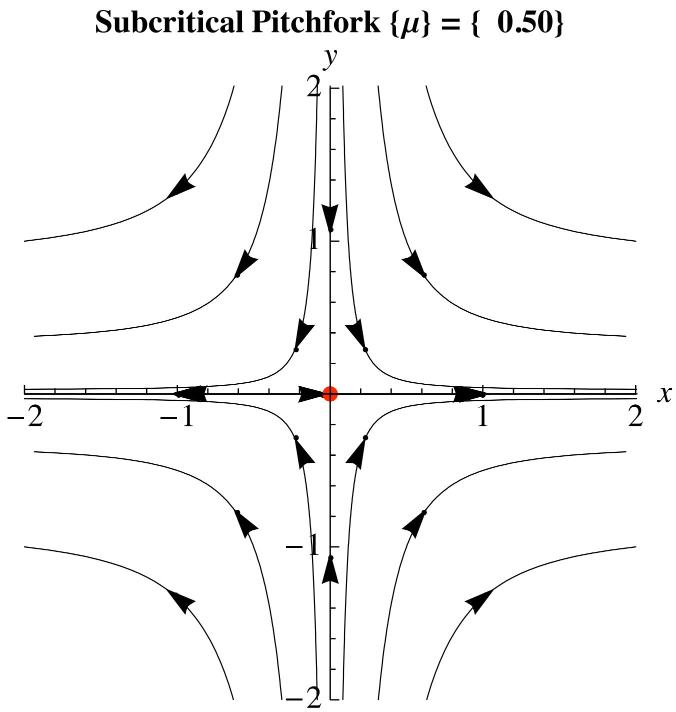


Subcritical Pitchfork $\{\mu\} = \{-0.25\}$




Subcritical Pitchfork {μ} = { 0.00}
y
2
1
−1
−2
x
−2
−1
1
2


Subcritical Pitchfork {μ} = { 0.25}
y
2
1
−1
−2
x
−2
−1
1
2



As our final task in this notebook, we will make a movie of the subcritical pitchfork bifurcation. We will concentrate the frames of the movie around $\mu = 0$. We make 41 frames at μ-intervals of 0.015, with μ running from -0.3 to 0.3. We remove the axes for a clearer view, and we use the colored dots for equilibria as before.

```
parmlist = Module[{ans, i}, ans = {};
  Do[ans = Append[ans, {0.015 * i}], {i, -20, 20}]; ans]
```

```
{{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21},
 {-0.195}, {-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12},
 {-0.105}, {-0.09}, {-0.075}, {-0.06}, {-0.045}, {-0.03},
 {-0.015}, {0}, {0.015}, {0.03}, {0.045}, {0.06}, {0.075}, {0.09},
 {0.105}, {0.12}, {0.135}, {0.15}, {0.165}, {0.18}, {0.195},
 {0.21}, {0.225}, {0.24}, {0.255}, {0.27}, {0.285}, {0.3}}
```

```
plrange = {{-2, 2}, {-2, 2}}; asprat = 1; axon  = False;

arrowflag = True; arrowvec = {1 / 2};

totdig = 5; decdig = 3;

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};
```

```
bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]
```

```
Bifurcation sequence for parmlist =
 {{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21}, {-0.195},
  {-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12}, {-0.105}, {-0.09}, {-0.075},
  {-0.06}, {-0.045}, {-0.03}, {-0.015}, {0}, {0.015}, {0.03}, {0.045},
  {0.06}, {0.075}, {0.09}, {0.105}, {0.12}, {0.135}, {0.15}, {0.165},
  {0.18}, {0.195}, {0.21}, {0.225}, {0.24}, {0.255}, {0.27}, {0.285}, {0.3}}
```

Subcritical Pitchfork {μ} = { −0.300}