1. Introduction

In this notebook, the fourth in a series of notebooks on bifurcations, we look at a simple example of a supercritical pitchfork bifurcation. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

2. Definition of the System

We consider the following system, depending on one parameter $a$:

\[
\dot{x} = \mu x - x^3, \quad \dot{y} = -y. 
\]

This system has one equilibrium for any $\mu \leq 0$, and three equilibria for $\mu > 0$. The bifurcation is $\mu = 0$, for which one equilibrium bifurcates into three as we increase $\mu$. We begin our analysis by defining the system for DynPac.

```mathematica
setstate[{x, y}]; setparm[{\mu}]; slopevec = {\mu x - x^3, -y};
sysname = "Supercritical Pitchfork";

eq1 = {0, 0}; eq2 = {\sqrt{\mu}, 0}; eq3 = {-\sqrt{\mu}, 0};
```

Let's look at the nature of the equilibria.

```mathematica
eigsysex{eq1}
{{{-1, \mu}, {{0, 1}, {1, 0}}}}

eigsysex{eq2}
{{{-1, -2 \mu}, {{0, 1}, {1, 0}}}}
```
Thus for \( \mu < 0 \), eq1 is a stable node. For \( \mu > 0 \), eq1 is a saddle, and both eq2 and eq3 are stable nodes. For \( \mu = 0 \), linearization is inconclusive, although in this case it is easy to integrate the equations directly and show that the equilibrium is stable. This kind of bifurcation is called a supercritical pitchfork. We may visualize this with a bifurcation diagram, showing the equilibria as functions of \( \mu \), with stable in solid, unstable in dashed.

```mathematica

eigsys[eq3]

\[ \{\{-1, -2 \mu\}, \{0, 1\}, \{1, 0\}\} \]
```

Imagine a gradual change of \( \mu \)-values from negative through zero to small and positive. For negative \( \mu \), the system will settle into the only attractor, the stable node at \( x = 0 \). It will stay here as we increase \( \mu \) slowly. When we just exceed \( \mu = 0 \), the equilibrium at \( x = 0 \) is unstable, and the system will jump to one of the stable equilibria \( \pm \sqrt{\mu} \). For small \( \mu \) these are both very close to \( x = 0 \), so that the system makes a small rather than a catastrophic jump. For a very different outcome, see the next notebook on subcritical pitchfork bifurcations.

Now we construct a short sequence of phase plots for different values of \( \mu \), for a given set of initial conditions. These will illustrate the bifurcation at \( \mu = 0 \). We will mark the equilibria by red dots for unstable,
blue dots for stable. We do this by constructing a graph refgraph which is then included in each picture of the bifurcation sequence.

refgraph := Module[{temp1, temp2, temp3, ans}, ptsize = 0.025;
  display = False;
  If[(First[parmval] > 0), (setcolor[{Red}]; temp1 = dots[{eq1}];
    setcolor[{Blue}]; temp2 = dots[{eq2}]; temp3 = dots[{eq3}];
    ans = {temp1, temp2, temp3}), (setcolor[{Blue}];
    ans = dots[{eq1}]); setcolor[{Black}]; ans]

  ε = 0.02;
  initset = {{2, 0}, {2, 1}, {2, 2}, {1, 2}, {0, 2}, {-1, 2},
    {-2, 2}, {-2, 1}, {-2, 0}, {-2, -1}, {-2, -2}, {-1, -2},
    {0, -2}, {1, -2}, {2, -2}, {2, -1}, {ε, 0}, {-ε, 0},
    {0.25, 0.25}, {0.25, -0.25}, {-0.25, 0.25}, {-0.25, -0.25}};

  prrange = {{-2, 2}, {-2, 2}}; asprat = 1;
  arrowflag = True; arrowvec = {1/2};
  t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;
  rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};

Now we choose a small number of $\mu$-values for a bifurcation sequence.

  parmlist = {{-0.5}, {-0.25}, {0.0}, {0.25}, {0.5}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]

Bifurcation sequence for parmlist = {{-0.5}, {-0.25}, {0.0}, {0.25}, {0.5}}
Supercritical Pitchfork \( \mu = \{-0.50\} \)

Supercritical Pitchfork \( \mu = \{-0.25\} \)
Supercritical Pitchfork $\{\mu\} = \{0.00\}$

Supercritical Pitchfork $\{\mu\} = \{0.25\}$
Supercritical Pitchfork \( \mu = \{ 0.50 \} \)

As our final task in this notebook, we will make a movie of the supercritical pitchfork bifurcation. We will concentrate the frames of the movie around \( \mu = 0 \). We make 41 frames at \( \mu \)-intervals of 0.015, with \( \mu \) running from -0.3 to 0.3. We remove the axes for a clearer view, and we use the colored dots for equilibria as before.

```mathematica
parmlist = Module[{ans, i}, ans = {};
   Do[ans = Append[ans, (0.015*i)], {i, -20, 20}]; ans]

{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21},
{-0.195}, {-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12},
{-0.105}, {-0.09}, {-0.075}, {-0.06}, {-0.045}, {-0.03},
{-0.015}, {0}, (0.015), {0.03}, (0.045), {0.06}, (0.075), (0.09),
(0.105), (0.12), (0.135), (0.15), (0.165), (0.18), (0.195),
(0.21), (0.225), (0.24), (0.255), (0.27), (0.285), (0.3)}

initset = {(2, 0), (2, 1), (2, 2), (1, 2), (0, 2), (-1, 2),
{-2, 2), (-2, 1), (-2, 0), (-2, -1), (-2, -2), (-1, -2),
(0, -2), (1, -2), (2, -2), (2, -1), (0, -1), (-2, 0),
(0.25, 0.25), (0.25, -0.25), (-0.25, 0.25), (-0.25, -0.25)};

plrange = {(-2, 2), (-2, 2)}; asprat = 1; axon = False;

arrowflag = True; arrowvec = {1/2};

totdig = 6; decdig = 3;

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {(-2.1, 2.1), (-2.1, 2.1)};
```
\[ \text{bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]} \]

Bifurcation sequence for parmlist =
\[
((-0.3), (-0.285), (-0.27), (-0.255), (-0.24), (-0.225), (-0.21), (-0.195), \\
(-0.18), (-0.165), (-0.15), (-0.135), (-0.12), (-0.105), (-0.09), (-0.075), \\
(-0.06), (-0.045), (-0.03), (-0.015), (0), (0.015), (0.03), (0.045), \\
(0.06), (0.075), (0.09), (0.105), (0.12), (0.135), (0.15), (0.165), \\
(0.18), (0.195), (0.21), (0.225), (0.24), (0.255), (0.27), (0.285), (0.3))
\]

Supercritical Pitchfork \( \{\mu\} = \{-0.300\} \)