

ME 406

Bifurcations IV

Supercritical Pitchfork Bifurcation

`sysid`

Mathematica 6.0.3, DynPac 11.01, 1/12/2009

`intreset; plotreset; imsize = 250;`

■ 1. Introduction

In this notebook, the fourth in a series of notebooks on bifurcations, we look at a simple example of a supercritical pitchfork bifurcation. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

■ 2. Definition of the System

We consider the following system, depending on one parameter a :

$$\dot{x} = \mu x - x^3, \quad \dot{y} = -y.$$

This system has one equilibrium for any $\mu \leq 0$, and three equilibria for $\mu > 0$. The bifurcation is $\mu = 0$, for which one equilibrium bifurcates into three as we increase μ . We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}]; slopevec = {μ x - x3, -y};  
sysname = "Supercritical Pitchfork";
```

```
eq1 = {0, 0}; eq2 = {√μ, 0}; eq3 = {-√μ, 0};
```

Let's look at the nature of the equilibria.

```
eigsys[eq1]
```

```
{{-1, μ}, {{0, 1}, {1, 0}}}
```

```
eigsys[eq2]
```

```
{{-1, -2 μ}, {{0, 1}, {1, 0}}}
```

```
eigsys[eq3]
```

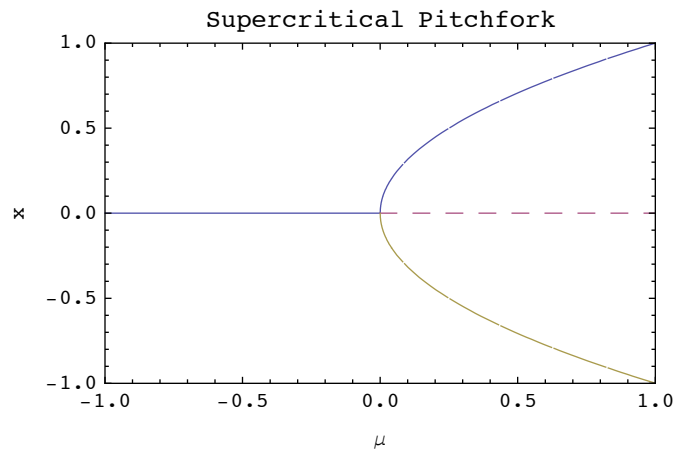
```
{{-1, -2 μ}, {{0, 1}, {1, 0}}}
```

Thus for $\mu < 0$, eq1 is a stable node. For $\mu > 0$, eq1 is a saddle, and both eq2 and eq3 are stable nodes. For $\mu = 0$, linearization is inconclusive, although in this case it is easy to integrate the equations directly and show that the equilibrium is stable. This kind of bifurcation is called a supercritical pitchfork. We may visualize this with a bifurcation diagram, showing the equilibria as functions of μ , with stable in solid, unstable in dashed.

```
plot1 = Plot[{0}, {μ, -1, 0},
  PlotRange → {{-1, 1}, {-1, 1}},
  PlotLabel → "Supercritical Pitchfork",
  FrameLabel → {"μ", "x"}, Axes → False,
  ImageSize → imsize, Frame → True, DisplayFunction → Identity];
```

```
plot2 = Plot[{√μ, 0, -√μ}, {μ, 0, 1},
  PlotRange → {{-1, 1}, {-1, 1}},
  PlotLabel → "Supercritical Pitchfork", Frame → True,
  FrameLabel → {"μ", "x"}, Axes → False,
  DisplayFunction → Identity, ImageSize → imsize, PlotStyle →
  {Dashing[{0.1, 0}], Dashing[{0.03, 0.03}], Dashing[{0.1, 0}]}];
```

```
Show[{plot1, plot2}]
```



Imagine a gradual change of μ -values from negative through zero to small and positive. For negative μ , the system will settle into the only attractor, the stable node at $x = 0$. It will stay here as we increase μ slowly. When we just exceed $\mu = 0$, the equilibrium at $x = 0$ is unstable, and the system will jump to one of the stable equilibria -- either $\sqrt{\mu}$ or $-\sqrt{\mu}$. For small μ these are both very close to $x = 0$, so that the system makes a small rather than a catastrophic jump. For a very different outcome, see the next notebook on subcritical pitchfork bifurcations.

Now we construct a short sequence of phase plots for different values of μ , for a given set of initial conditions. These will illustrate the bifurcation at $\mu = 0$. We will mark the equilibria by red dots for unstable,

blue dots for stable. We do this by constructing a graph refgraph which is then included in each picture of the bifurcation sequence.

```

refgraph := Module[{temp1, temp2, temp3, ans}, ptsize = 0.025;
  display = False;
  If[(First[parmval] > 0), (setcolor[{Red}]; temp1 = dots[{eq1}];
    setcolor[{Blue}]; temp2 = dots[{eq2}]; temp3 = dots[{eq3}];
    ans = {temp1, temp2, temp3}), (setcolor[{Blue}];
    ans = dots[{eq1}])]; setcolor[{Black}]; ans]

ε = 0.02;

initset = {{2, 0}, {2, 1}, {2, 2}, {1, 2}, {0, 2}, {-1, 2},
  {-2, 2}, {-2, 1}, {-2, 0}, {-2, -1}, {-2, -2}, {-1, -2},
  {0, -2}, {1, -2}, {2, -2}, {2, -1}, {ε, 0}, {-ε, 0},
  {0.25, 0.25}, {0.25, -0.25}, {-0.25, 0.25}, {-0.25, -0.25}};

plrange = {{-2, 2}, {-2, 2}}; asprat = 1;

arrowflag = True; arrowvec = {1 / 2};

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};

```

Now we choose a small number of μ -values for a bifurcation sequence.

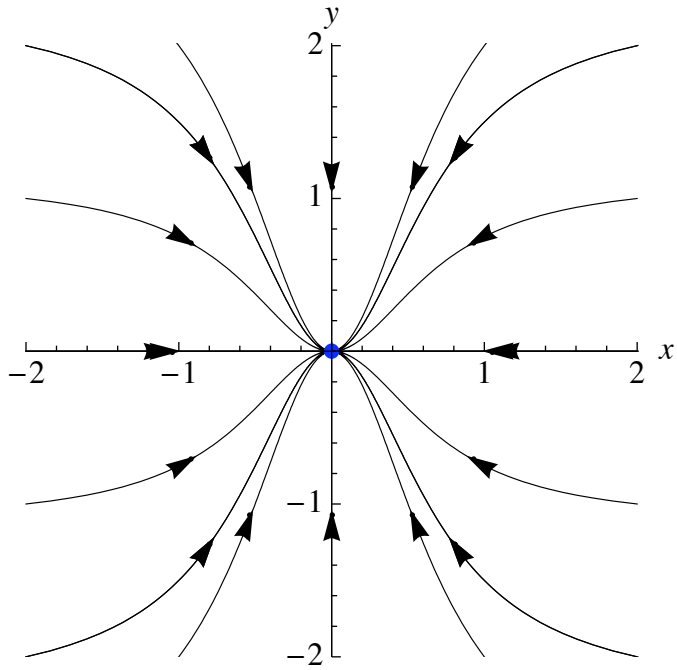
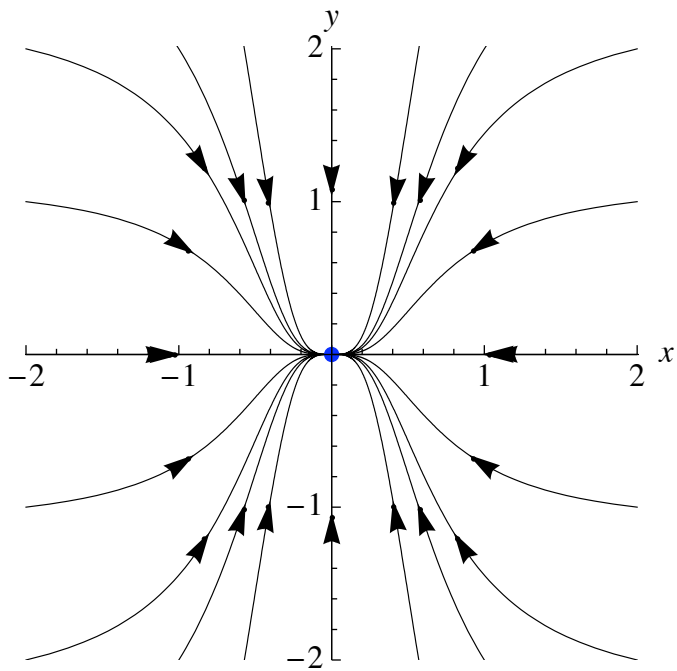
```

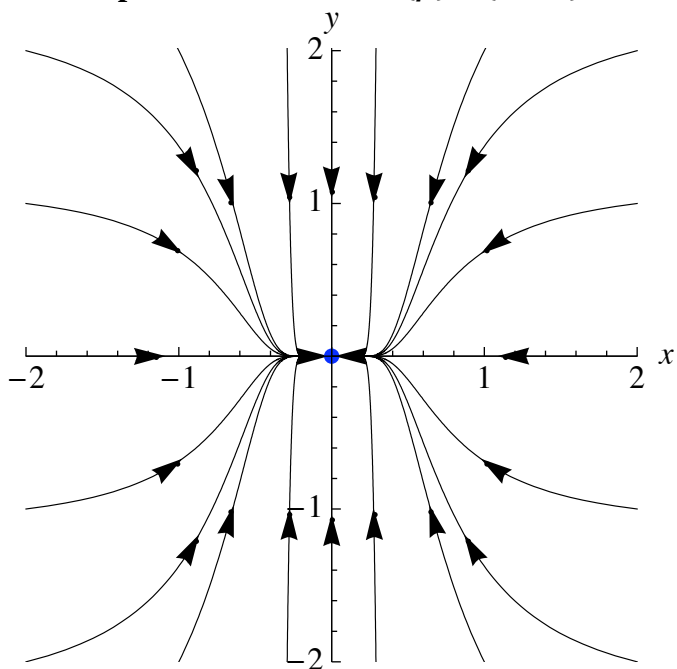
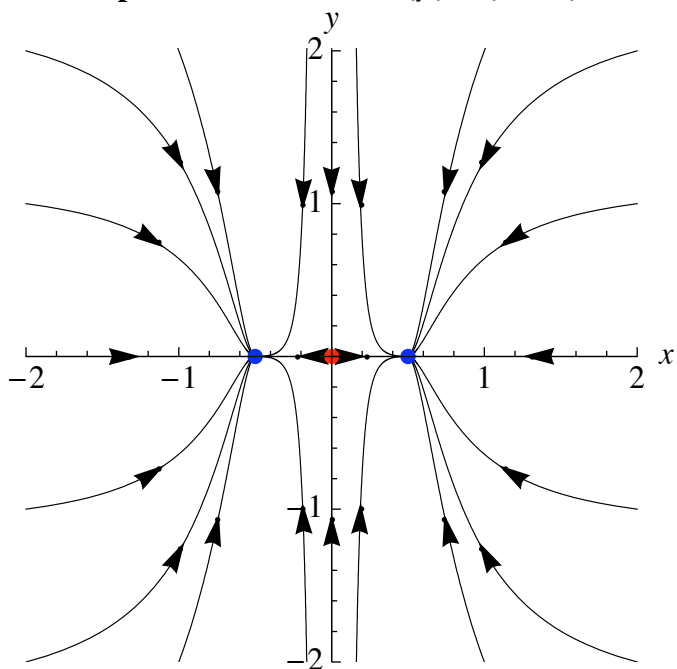
parmlist = {{-0.5}, {-0.25}, {0.0}, {0.25}, {0.5}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]

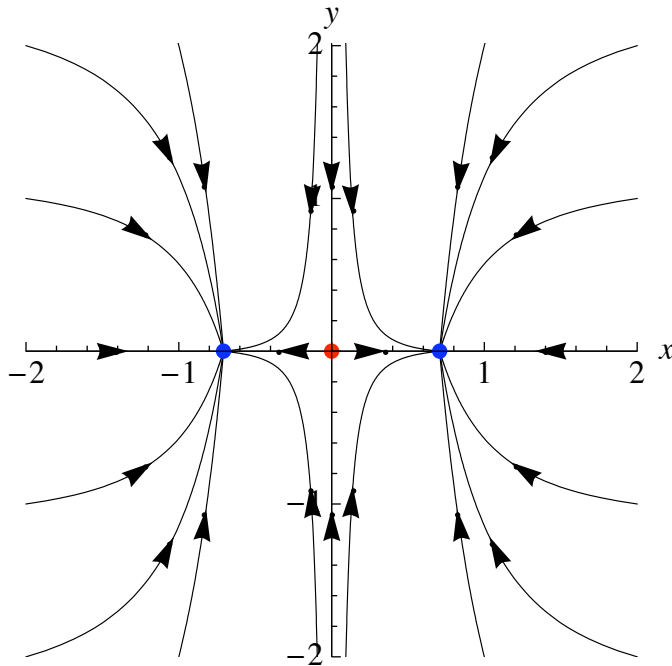
Bifurcation sequence for parmlist = {{-0.5}, {-0.25}, {0.}, {0.25}, {0.5}}

```

Supercritical Pitchfork $\{\mu\} = \{-0.50\}$ **Supercritical Pitchfork $\{\mu\} = \{-0.25\}$** 

Supercritical Pitchfork $\{\mu\} = \{ 0.00\}$ **Supercritical Pitchfork $\{\mu\} = \{ 0.25\}$** 

Supercritical Pitchfork $\{\mu\} = \{0.50\}$



As our final task in this notebook, we will make a movie of the supercritical pitchfork bifurcation. We will concentrate the frames of the movie around $\mu = 0$. We make 41 frames at μ -intervals of 0.015, with μ running from -0.3 to 0.3. We remove the axes for a clearer view, and we use the colored dots for equilibria as before.

```

parmlist = Module[{ans, i}, ans = {};
  Do[ans = Append[ans, {0.015 * i}], {i, -20, 20}]; ans]

{{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21},
{-0.195}, {-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12},
{-0.105}, {-0.09}, {-0.075}, {-0.06}, {-0.045}, {-0.03},
{-0.015}, {0}, {0.015}, {0.03}, {0.045}, {0.06}, {0.075}, {0.09},
{0.105}, {0.12}, {0.135}, {0.15}, {0.165}, {0.18}, {0.195},
{0.21}, {0.225}, {0.24}, {0.255}, {0.27}, {0.285}, {0.3}}

initset = {{2, 0}, {2, 1}, {2, 2}, {1, 2}, {0, 2}, {-1, 2},
{-2, 2}, {-2, 1}, {-2, 0}, {-2, -1}, {-2, -2}, {-1, -2},
{0, -2}, {1, -2}, {2, -2}, {2, -1}, {ε, 0}, {-ε, 0},
{0.25, 0.25}, {0.25, -0.25}, {-0.25, 0.25}, {-0.25, -0.25}};

plrange = {{-2, 2}, {-2, 2}}; asprat = 1; axon = False;

arrowflag = True; arrowvec = {1/2};

totdig = 6; decdig = 3;

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};

```

```
bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]
```

Bifurcation sequence for parmlist =

```
{{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21}, {-0.195},  
{-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12}, {-0.105}, {-0.09}, {-0.075},  
{-0.06}, {-0.045}, {-0.03}, {-0.015}, {0}, {0.015}, {0.03}, {0.045},  
{0.06}, {0.075}, {0.09}, {0.105}, {0.12}, {0.135}, {0.15}, {0.165},  
{0.18}, {0.195}, {0.21}, {0.225}, {0.24}, {0.255}, {0.27}, {0.285}, {0.3}}
```

Supercritical Pitchfork $\{\mu\} = \{-0.300\}$

