ME 406
Bifurcations II
Saddle-Node Bifurcation

sysid
Mathematica 6.0.3, DynPac 11.01, 1/12/2009

intreset; plotreset; imsize = 250;

1. Introduction

In this notebook, the second in a series of notebooks on bifurcations, we look at simple examples of a saddle-node bifurcation. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

2. Definition of the System

We consider the following system, depending on one parameter $a$:

$$\dot{x} = \mu - x^2, \quad \dot{y} = -y.$$

This system has two equilibria for $\mu > 0$, one for $\mu = 0$, and none for $\mu < 0$. We begin our analysis by defining the system for DynPac.

```mathematica
setstate[{x, y}]; setparm[\{\mu\}]; slopevec = \{\mu - x^2, -y\};
sysname = "Saddle-Node Bifurcation";
```

We give a names to the equilibrium points for $\mu > 0$.

```mathematica
eq1 = \{Sqrt[\mu], 0\}; eq2 = \{-Sqrt[\mu], 0\};
```

We see that as $\mu \to 0+$, the two equilibria coalesce. For $\mu < 0$, there are no equilibria. One way of describing this sequence of events is that as $\mu \to 0+$, the equilibria collide and annihilate one another.

Let's look at the nature of the equilibria for positive $\mu$.

```mathematica
eigs[eq1]
```

$$\left\{ \{-1, -2\sqrt{\mu}\}, \{\{0, 1\}, \{1, 0\}\} \right\}$$
eigsys[eq2]
\[ \{\{-1, 2 \sqrt{\mu}\}, \{(0, 1), (1, 0)\}\} \]

Thus eq1 is a stable node and eq2 is a saddle. A conventional way to represent this situation graphically is to plot the x-positions of the equilibria as a function of the parameter \( \mu \), with the stable equilibrium solid and the unstable equilibrium dashed.

```
Plot[\[\{\sqrt{\mu}, -\sqrt{\mu}\}, \{\mu, 0, 1\}, PlotRange \rightarrow \{-1, 1\}, \{-1.1, 1.1\}\],
PlotLabel \rightarrow "Saddle-Node Bifurcation",
AxesLabel \rightarrow \"x\", ImageSize \rightarrow imsize,
PlotStyle \rightarrow \{Dashing[\{0.1\}], Dashing[\{0.03, 0.03\}]\}]
```

Now we will construct phase plots for various values of the parameter \( \mu \). We first construct a short sequence which will show the essential features, and then we construct a long sequence suitable for a movie. We choose a plotting window of \{\{-2,2\},\{-2,2\}\}, and a set of initial conditions attached to points on the window, plus several initial conditions near the equilibria.

\[ f[\mu_] := \text{If}[\mu > 0, \text{Sqrt}[\mu], 0] \]
\[ \varepsilon = 0.02; \]
\[ \text{initset} = \{(2, 0), (2, -1.5), (2, 1.5), (0, 0),
\{2, 0\}, (1.5, 2), (-0.5, 2), (1.5, -2), (-0.5, -2),
\{-0.5, 2\}, \{f[\mu], \varepsilon\}, \{-f[\mu], \varepsilon\}, \{-f[\mu], -\varepsilon\}]; \]
\[ \text{plrange} = \{(\{-2, 2\}, \{-2, 2\})\}; \]
\[ \text{arrowflag} = \text{True}; \text{arrowvec} = \{1/2\}; \]
\[ \text{t0} = 0.0; h = 0.02; \text{nsteps} = 800; \text{bothdirflag} = \text{True}; \]
\[ \text{rangeflag} = \text{True}; \text{ranger} = \{(\{-2.1, 2.1\}, \{-2.1, 2.1\})\}; \]

Now we choose a small number of \( \mu \)-values for a bifurcation sequence.
parmlist = {(0.5), (0.25), (0.0), (-0.25), (-0.5)};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist]

Bifurcation sequence for parmlist = {(0.5), (0.25), (0.), (-0.25), (-0.5)}

Saddle–Node Bifurcation \{\mu\} = \{ 0.50\}

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Saddle–Node Bifurcation \( \{\mu\} = \{0.00\} \)

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Saddle–Node Bifurcation \( \{ \mu \} = \{-0.50\} \)

The equilibrium for \( \mu = 0 \) is an interesting composite. It looks like a stable node on the right, and like a saddle on the left. For \( \mu = -0.25 \), there is a kind of "ghost" of the vanished equilibria. We can see that more directly by looking at the time-dependence of a solution for a small negative value of \( \mu \). We do that for \( \mu = -0.05 \).

\[
\text{parmval} = \{-0.05\};
\]
\[
\text{t0} = 0.0; \text{h} = 0.02; \text{nsteps} = 500; \text{initvec} = \{1, 0\}; \text{bothdirflag} = \text{False};
\]
\[
\text{ghostsol} = \text{integrate}[\text{initvec}, \text{t0}, \text{h}, \text{nsteps}];
\]
\[
\text{plrange} = \{(0, 10), (-2, 1)\}; \text{asprat} = 1.0;
\]
We see that the system state lingers around \( x = 0 \) for some time. Put another way, the system takes a relatively long time to pass the site of the former equilibrium state.

As our final task in this notebook, we will make a movie of the saddle-node bifurcation. We will concentrate the frames of the movie around \( \mu = 0 \). We make 21 frames at \( \mu \)-intervals of 0.01, with \( \mu \) running from 0.1 to -0.1. We remove the axes for a clearer view, and we place a red dot on the saddle, a blue dot on the stable node. The function \texttt{refgraph} below defines the dot graphs.

\[
\texttt{refgraph} := \text{Module}[[\texttt{temp1}, \texttt{temp2}],
\text{If}[\mu < 0, (\texttt{ans} = \{\}), (\texttt{ptsize} = 0.02; \texttt{setcolor}[\{\texttt{Blue}\}];
\texttt{temp1} = \texttt{dots}[\{\texttt{eq1}\}]; \texttt{setcolor}[\{\texttt{Red}\}]; \texttt{temp2} = \texttt{dots}[\{\texttt{eq2}\}];
\texttt{setcolor}[\{\texttt{Black}\}]; \texttt{ans} = \{\texttt{temp1, temp2}\}])
\]

\[
\texttt{parmlist} := \text{Module}[[\texttt{ans}, \texttt{i}], \texttt{ans} = \{\}]
\text{Do}[\texttt{ans} = \text{Append}[\texttt{ans}, \{0.01*\texttt{i}\}], \{\texttt{i}, 10, -10, -1\}]; \texttt{ans}
\]

\[
\texttt{initset} = \{(2, 0), (2, -1.5), (2, 1.5), (0, 0),
(\text{-}2, 0), (1.5, 2), (-0.5, 2), (1.5, -2), (-0.5, -2),
(-0.5, 2), \{\text{f}[\mu], \text{e}\}, \{\text{f}[\mu], -\text{e}\}, \{-\text{f}[\mu], \text{e}\}, \{-\text{f}[\mu], -\text{e}\})
\]

\[
\texttt{plrange} = \{\text{-}2, 2\}; \texttt{asprat} = 1; \texttt{axon} = \text{False};
\]

\[
\texttt{arrowflag} = \text{True}; \texttt{arrowvec} = \{1/2\};
\]

\[
\texttt{t0} = 0.0; \texttt{h} = 0.02; \texttt{nsteps} = 800; \texttt{bothdirflag} = \text{True};
\]
rangeflag = True; ranger = {(-2.1, 2.1), (-2.1, 2.1)};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]

Bifurcation sequence for parmlist =
{(0.1), (0.09), (0.08), (0.07), (0.06), (0.05), (0.04),
 (0.03), (0.02), (0.01), (0), (-0.01), (-0.02), (-0.03), (-0.04),
 (-0.05), (-0.06), (-0.07), (-0.08), (-0.09), (-0.1)}

Saddle–Node Bifurcation \{ \mu \} = \{ 0.10\}