ME 406
Bifurcations II
Saddle-Node Bifurcation

```
sysid
```

```
Mathematica  6.0.3, DynPac 11.01, 1/12/2009
```

```
intreset; plotreset; imsize = 250;
```

1. Introduction

In this notebook, the second in a series of notebooks on bifurcations, we look at simple examples of a saddle-node bifurcation. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

2. Definition of the System

We consider the following system, depending on one parameter a:

$$\dot{x} = \mu - x^2 , \quad \dot{y} = -y \; .$$

This system has two equilibria for $\mu > 0$, one for $\mu = 0$, and none for $\mu < 0$. We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}]; slopevec = {μ - x^2, - y};
sysname = "Saddle-Node Bifurcation";
```

We give a names to the equilibrium points for $\mu > 0$.

```
eq1 = {Sqrt[μ], 0}; eq2 = {-Sqrt[μ], 0};
```

We see that as $\mu \to 0+$, the two equilibria coalesce. For $\mu < 0$, there are no equilibria. One way of describing this sequence of events is that as $\mu \to 0+$, the equilibria collide and annihilate one another.

Let's look at the nature of the equilibria for positive μ.

```
eigsys[eq1]
```

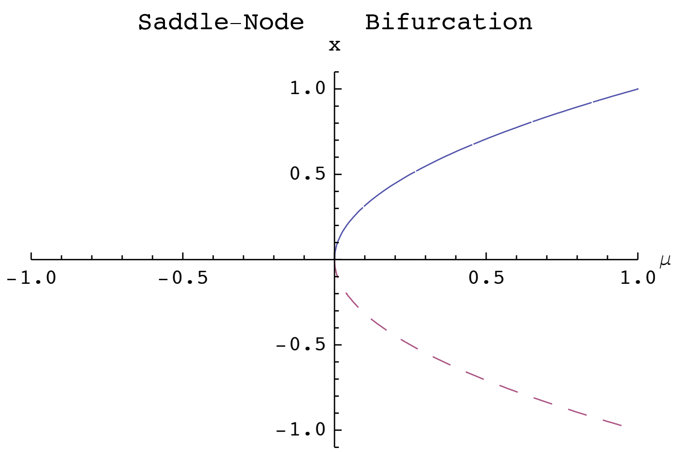
$$\left\{\left\{-1, -2\sqrt{\mu}\right\}, \{\{0, 1\}, \{1, 0\}\}\right\}$$

```
eigsys[eq2]
```

$\{\{-1, 2\sqrt{\mu}\}, \{\{0, 1\}, \{1, 0\}\}\}$

Thus eq1 is a stable node and eq2 is a saddle. A conventional way to represent this situation graphically is to plot the *x*-positions of the equilibria as a function of the parameter μ, with the stable equilibrium solid and the unstable equilibrium dashed.

```
Plot[{√μ, -√μ}, {μ, 0, 1}, PlotRange → {{-1, 1}, {-1.1, 1.1}},
 PlotLabel → "Saddle-Node    Bifurcation",
 AxesLabel → {"μ", "x"}, ImageSize → imsize,
 PlotStyle → {Dashing[{0.1, 0}], Dashing[{0.03, 0.03}]}]
```



Now we will construct phase plots for various values of the parmeter μ. We first construct a short sequence which will show the essential features, and then we construct a long sequence suitable for a movie. We choose a plotting window of {{-2,2},{-2,2}}, and a set of initial conditions attached to points on the window, plus several initial conditions near the equilibria.

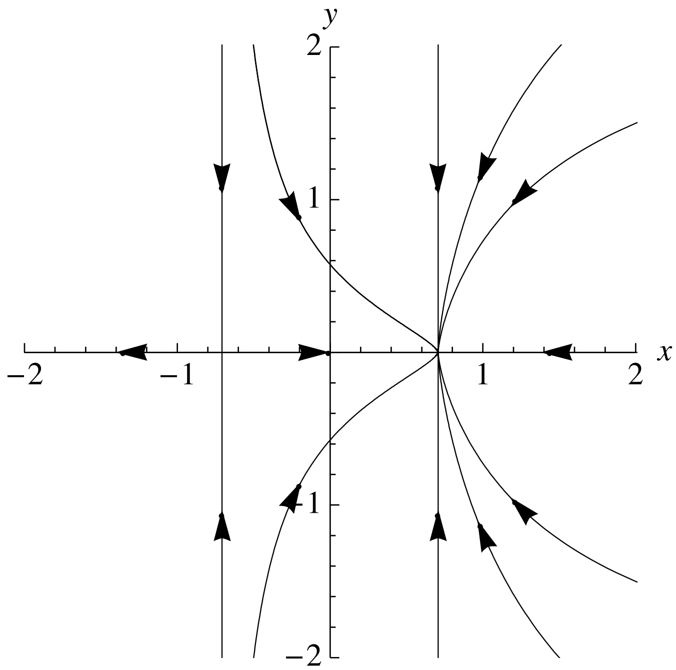
```
f[μ_] := If[μ > 0, Sqrt[μ], 0]

ε = 0.02;

initset = {{2, 0}, {2, -1.5}, {2, 1.5}, {0, 0},
   {-2, 0}, {1.5, 2}, {-0.5, 2}, {1.5, -2}, {-0.5, -2},
   {-0.5, 2}, {f[μ], ε}, {f[μ], -ε}, {-f[μ], ε}, {-f[μ], -ε}};

plrange = {{-2, 2}, {-2, 2}}; asprat = 1;

arrowflag = True; arrowvec = {1 / 2};

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};
```

Now we choose a small number of μ-values for a bifurcation sequence.

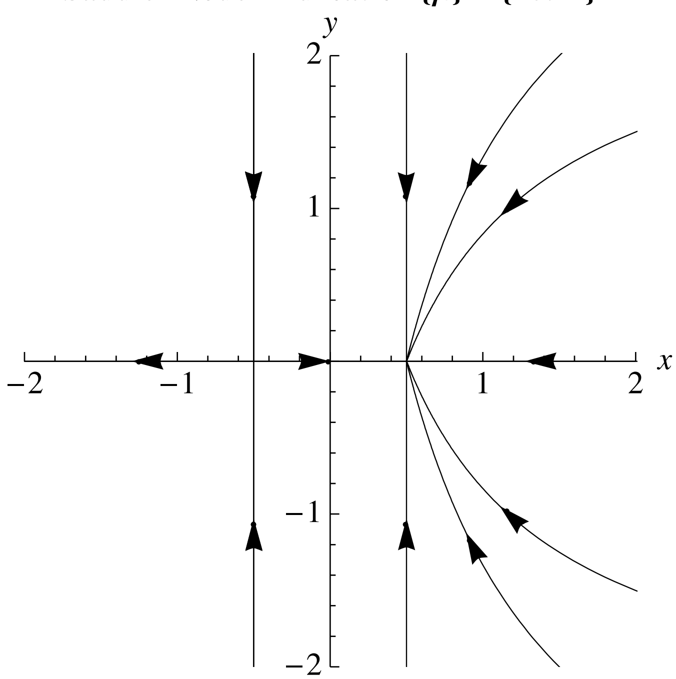
```
parmlist = {{0.5}, {0.25}, {0.0}, {-0.25}, {-0.5}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist]
```

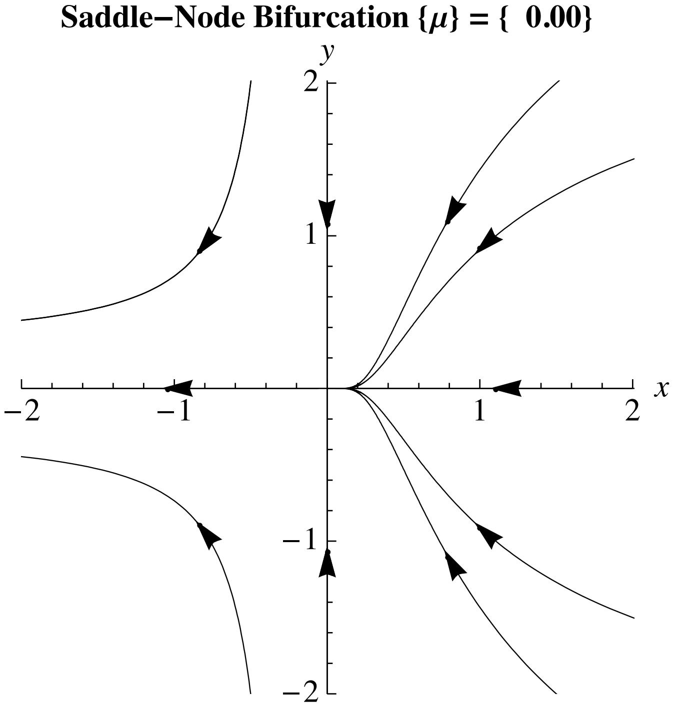
Bifurcation sequence for parmlist = {{0.5}, {0.25}, {0.}, {-0.25}, {-0.5}}

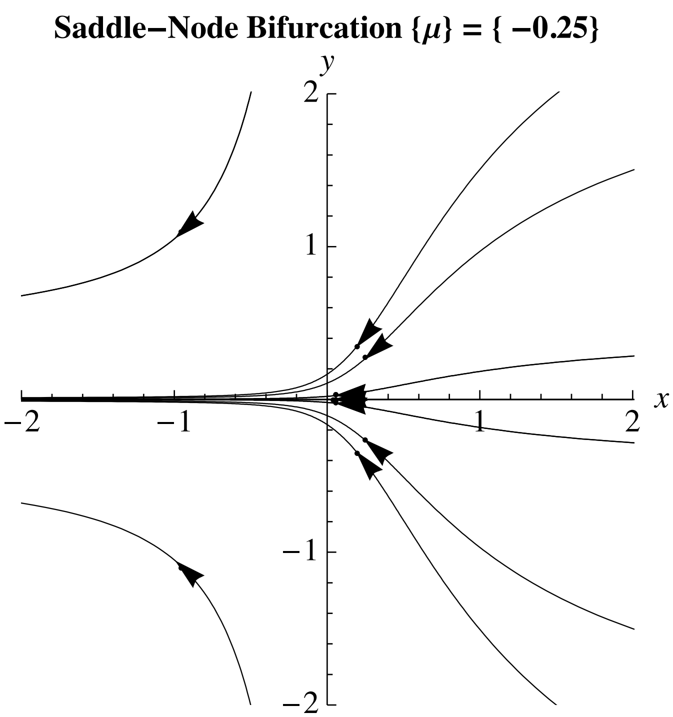
Saddle−Node Bifurcation {μ} = { 0.50}

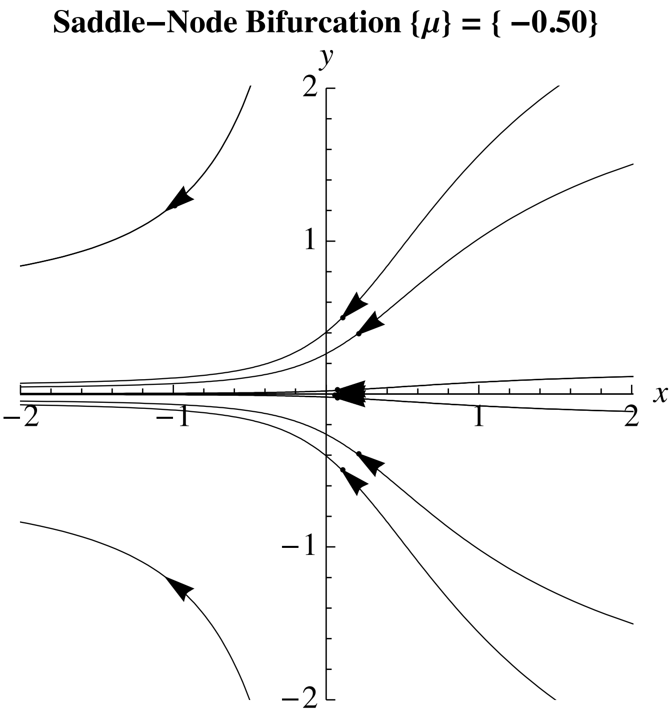


Saddle−Node Bifurcation {μ} = { 0.25}




Saddle−Node Bifurcation {μ} = { 0.00}
y
x
2
1
−1
−2
−2
−1
1
2

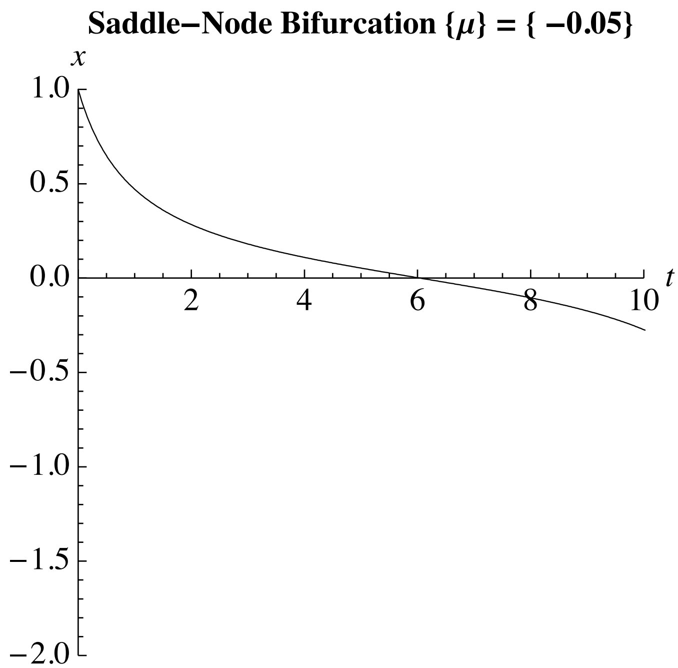

Saddle−Node Bifurcation {μ} = { −0.25}
y
x
2
1
−1
−2
−2
−1
1
2



The equilibrium for $\mu = 0$ is an interesting composite. It looks like a stable node on the right, and like a saddle on the left. For $\mu = -0.25$, there is a kind of "ghost" of the vanished equilibria. We can see that more directly by looking at the time-dependence of a solution for a small negative value of μ. We do that for $\mu = -0.05$.

```
parmval = {-0.05};

t0 = 0.0; h = 0.02; nsteps = 500; initvec = {1, 0}; bothdirflag = False;

ghostsol = integrate[initvec, t0, h, nsteps];

plrange = {{0, 10}, {-2, 1}}; asprat = 1.0;
```

```
ghostplot = timeplot[ghostsol, 1]
```



We see that the system state lingers around $x = 0$ for some time. Put another way, the system takes a relatively long time to pass the site of the former equilibrium state.

As our final task in this notebook, we will make a movie of the saddle-node bifurcation. We will concentrate the frames of the movie around $\mu = 0$. We make 21 frames at μ-intervals of 0.01, with μ running from 0.1 to -0.1. We remove the axes for a clearer view, and we place a red dot on the saddle, a blue dot on the stable node. The function refgraph below defines the dot graphs.

```
refgraph := Module[{temp1, temp2},
  If[μ < 0, (ans = {}), (ptsize = 0.02; setcolor[{Blue}];
    temp1 = dots[{eq1}]; setcolor[{Red}]; temp2 = dots[{eq2}];
    setcolor[{Black}]; ans = {temp1, temp2})]]
```

```
parmlist = Module[{ans, i}, ans = {};
  Do[ans = Append[ans, {0.01 * i}], {i, 10, -10, -1}]; ans]
```

```
{{0.1}, {0.09}, {0.08}, {0.07}, {0.06}, {0.05}, {0.04},
 {0.03}, {0.02}, {0.01}, {0}, {-0.01}, {-0.02}, {-0.03},
 {-0.04}, {-0.05}, {-0.06}, {-0.07}, {-0.08}, {-0.09}, {-0.1}}
```

```
initset = {{2, 0}, {2, -1.5}, {2, 1.5}, {0, 0},
   {-2, 0}, {1.5, 2}, {-0.5, 2}, {1.5, -2}, {-0.5, -2},
   {-0.5, 2}, {f[μ], ε}, {f[μ], -ε}, {-f[μ], ε}, {-f[μ], -ε}};
```

```
plrange = {{-2, 2}, {-2, 2}}; asprat = 1; axon = False;
```

```
arrowflag = True; arrowvec = {1 / 2};
```

```
t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;
```

```
rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph]
```

```
Bifurcation sequence for parmlist =
 {{0.1}, {0.09}, {0.08}, {0.07}, {0.06}, {0.05}, {0.04},
  {0.03}, {0.02}, {0.01}, {0}, {-0.01}, {-0.02}, {-0.03}, {-0.04},
  {-0.05}, {-0.06}, {-0.07}, {-0.08}, {-0.09}, {-0.1}}
```

Saddle−Node Bifurcation {μ} = { 0.10}