

ME 406

Orbital Stability

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■ 1. Introduction

In this notebook we develop some examples which illustrate the concept of orbital stability. We begin by reviewing the "standard" definition of stability, which is usually called Liapunov stability. The system under consideration is an autonomous set of n first order differential equations:

$$\dot{X} = F(X) \quad ,$$

where X is the state vector and F is the slope function.

Here is the definition of Liapunov stability. Given a solution $\mathbf{X}^*(t)$ with a given initial value $\mathbf{X}^*(t_0)$, we say that \mathbf{X}^* is stable if, given any $\epsilon > 0$, we can find a $\delta > 0$ such that for any solution $\mathbf{X}(t)$ satisfying $\|\mathbf{X}^*(t_0) - \mathbf{X}(t_0)\| < \delta$, it is true that $\|\mathbf{X}^*(t) - \mathbf{X}(t)\| < \epsilon$ for all $t \geq t_0$. We call a stable solution \mathbf{X}^* strictly stable if there exists an $\eta > 0$ such that $\|\mathbf{X}^*(t_0) - \mathbf{X}(t_0)\| < \eta$ implies that $\|\mathbf{X}^*(t) - \mathbf{X}(t)\|$ goes to zero as $t \rightarrow \infty$.

This way of defining stability means that neighboring solutions evaluated at the same time must remain neighboring. We shall see below that for periodic solutions this is in general too restrictive a definition of stability. A more appropriate definition of stability for such systems will be given after our first example.

■ 2. Example 1 - A Nonlinear Center

We consider Duffing's equation with a hard spring, a parameter a , and no damping: $\ddot{x} + x + a x^3 = 0$.

We define the system for DynPac.

```
setstate[{x, y}]; setparm[{a}];  
slopevec = {y, -x - a x^3}; sysname = "Duffing";
```

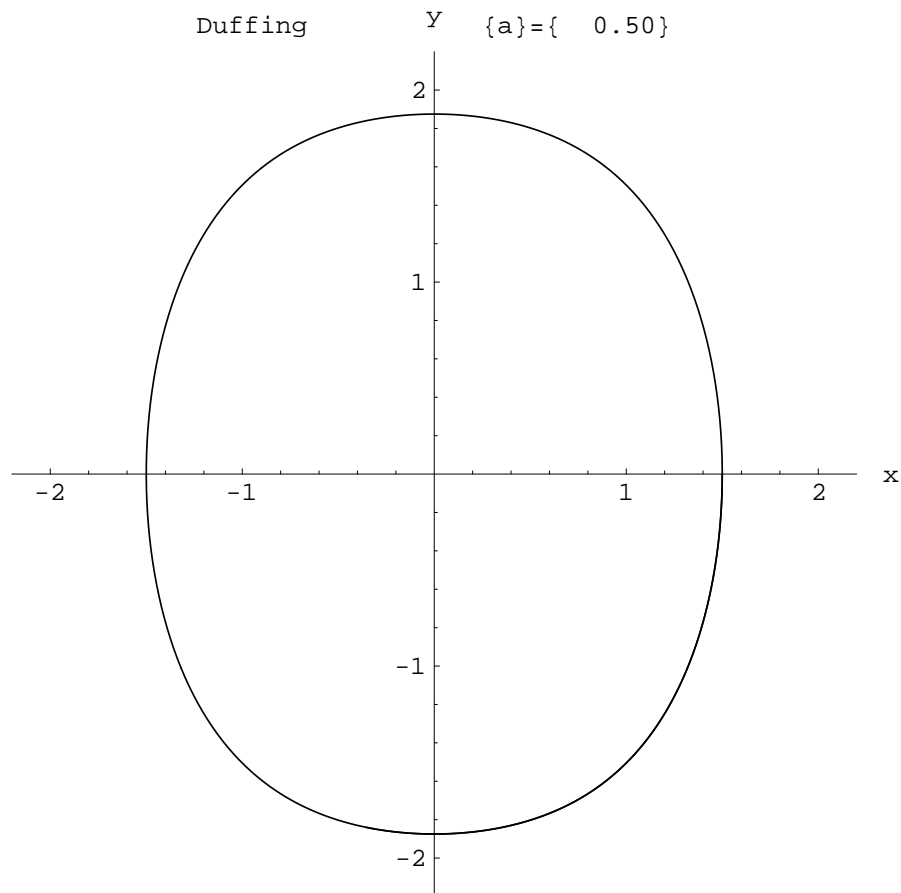
We set a parameter value of 0.5.

```
parmval = {0.5};
```

As we already know, all of the solutions of this equation for $a \geq 0$ are periodic. As an example, let's find and plot the solution with initial conditions (1.5, 0).

```
t0 = 0.0; h = 0.02; nsteps = 300; initvec = {1.5, 0};  
sol1 = integrate[initvec, t0, h, nsteps];  
asprat = 1.0; plrange = {{-2.2, 2.2}, {-2.2, 2.2}};
```

```
labshift = 10;  
graph1 = phaser[sol1];
```



We find the period of the solution:

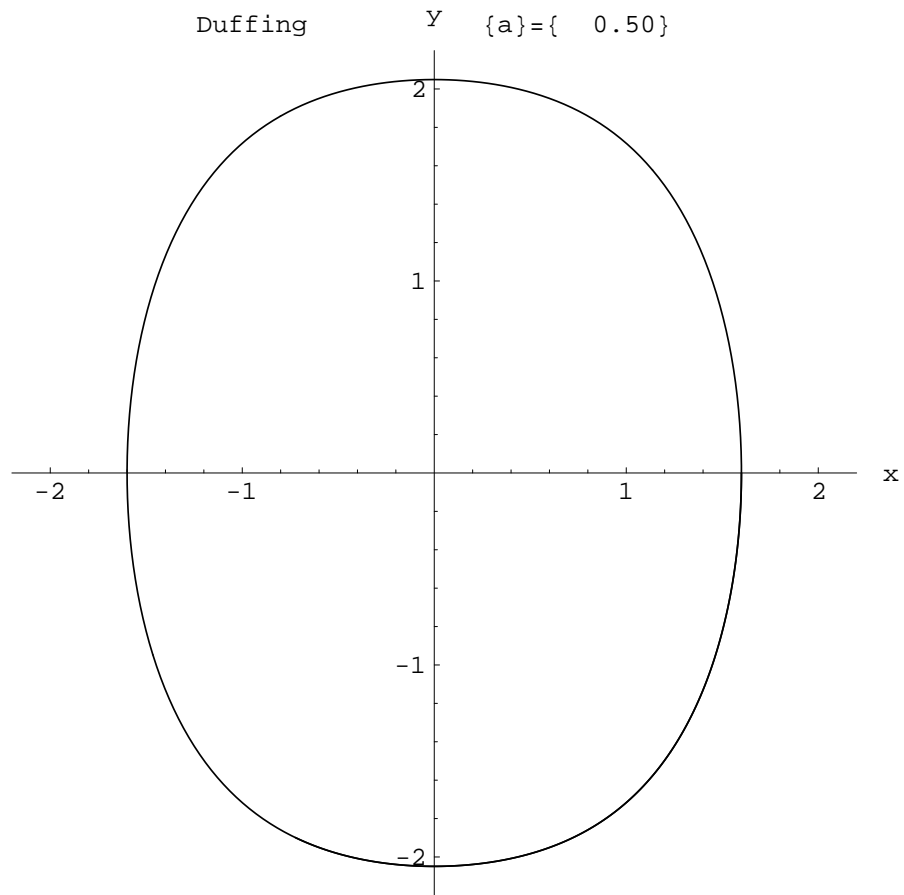
```
period[sol1]
```

4.64

Now consider a nearby solution, obtained by altering the initial conditions slightly.

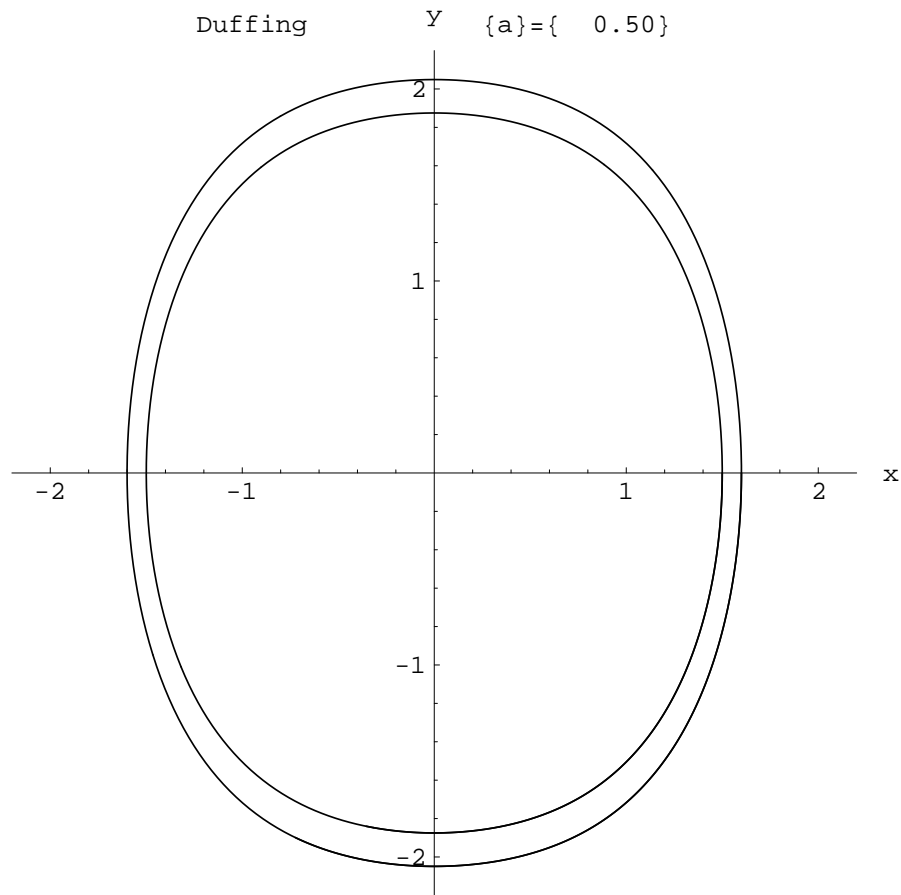
```
initvec = {1.6, 0}; sol2 = integrate[initvec, t0, h, nsteps];
```

```
graph2 = phaser[sol2];
```



We show the two graphs together.

```
graph3 = show[graph1, graph2];
```



The orbits are everywhere close, and we could have made them even closer had we chosen the initial values of x to be even closer. This is a situation which would be reasonable to call stable. However, by our earlier definition of stability, these solutions are unstable. It is because of the period. For the second solution we have

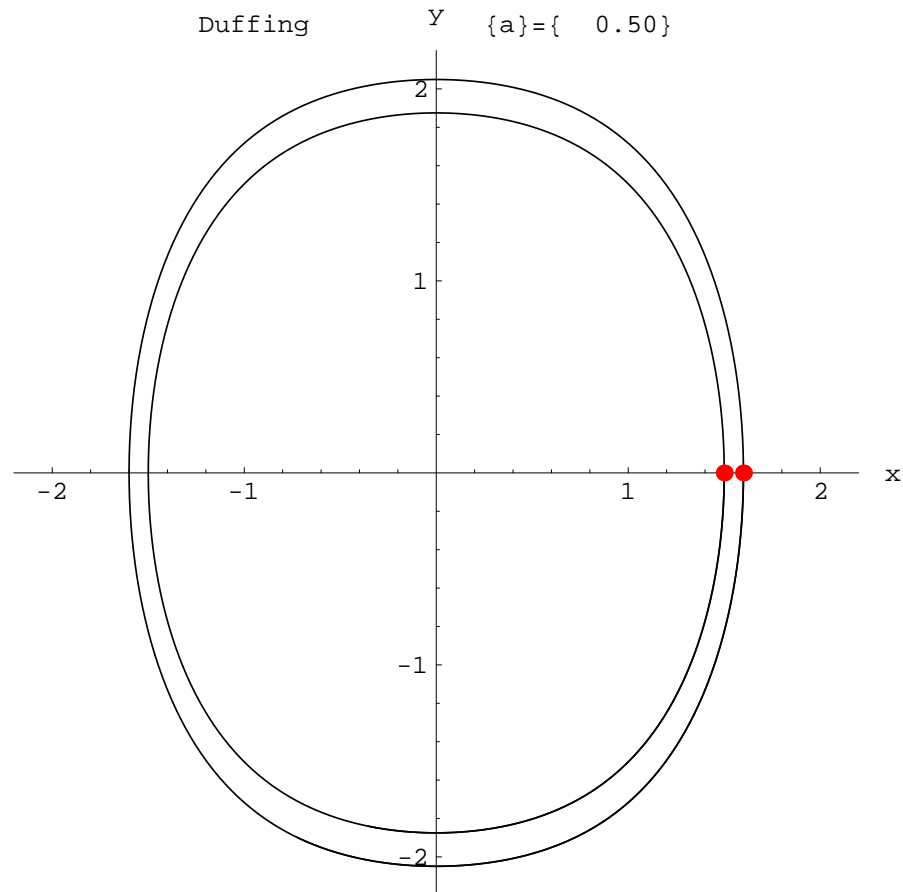
```
period[sol2]
```

```
4.52
```

Thus the periods differ slightly. Points on neighboring orbits will not remain close, because they go around the orbits in different times. Let's make a movie to illustrate that more graphically. We use the function `phaseseq`. Double click on the graph below to start the movie. In the printed version of this notebook, only the first frame of the movie is shown.

```
setcolor[{Red, Black, Black}];
```

```
phaseseq[{{1.5, 0}, {1.6, 0}}, t0, h, 10, 160, 1, 2, graph3];
```



The situation just illustrated clearly calls for an alternative definition of stability. Even though the two solutions at any given time do not remain close, the two orbits do remain close in space. The concept of orbital stability is appropriate to describe this situation. A solution \mathbf{X}^* with orbit Γ^* is said to be orbitally stable if given any $\epsilon > 0$, we can find a δ such that for any solution \mathbf{X} starting at t_0 a distance less than δ from Γ^* , it is true that for all $t \geq t_0$, the distance between the solution \mathbf{X} and Γ^* remains less than ϵ . Although this is intuitively clear, one needs to define precisely the concept of the distance between a solution and an orbit. This is done, for example, in **Nonlinear Ordinary Differential Equations**, by D.W. Jordn and P. Smith, second edition, Oxford University Press, 1987, Chapter 8, or **Dynamics and Bifurcations**, J. Hale and H. Koçak, Springer-Verlag, 1991, Chapter 11. It is clear that by this new definition, solutions of Duffing's equation are stable.

■ 3. Limit Cycles

With limit cycles, the concept of Liapunov stability may apply, but not strict Liapunov stability. To see that, consider two initial points both on the cycle. When the points have gone through one cycle, they will be exactly the same distance apart as when they started, so the distance between the two solutions will not go to zero as $t \rightarrow \infty$. With slight variations, the same situation is true for one initial point on the cycle and one just off the cycle. The orbit of the initial point just off the cycle will approach the cycle for large t , but in general there will always be a phase difference between the two points. Hale and Koçak (Chapter 11 -- cited above) give an excellent discussion of these concepts, along with definitions of three levels of orbital stability. We will content ourselves here with constructing two movies which illustrate some of the concepts for stable limit cycles. We use the van der Pol cycle. We will first construct a movie showing the motion of two phase points which start with a small separation on the limit cycle. Then we will construct a movie showing one phase point starting on the cycle, and one phase point starting just inside the cycle.

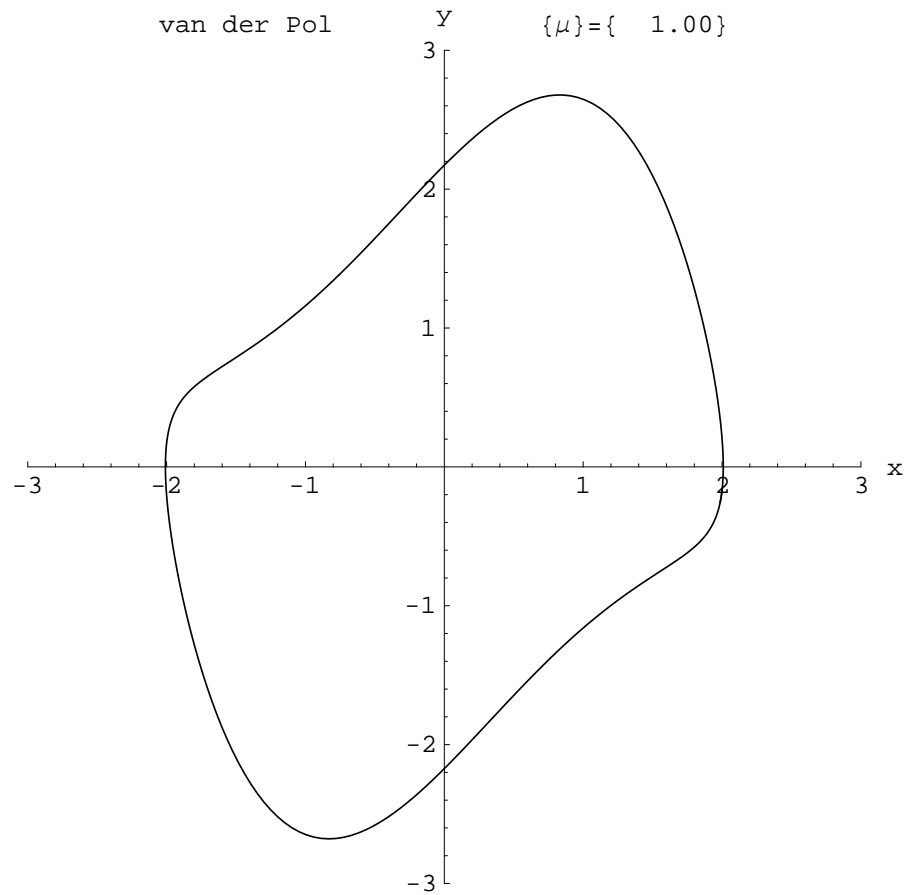
We begin by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}]; slopevec = {y, -x - μ * (x2 - 1) * y};  
sysname = "van der Pol"; parmval = {1};
```

We construct a graph of the limit cycle.

```
initvec = {2, 0}; t0 = 0.0; h = 0.02; nsteps = 337;  
sol4 = limcyc[initvec, t0, h, nsteps];  
plrange = {{-3, 3}, {-3, 3}}; asprat = 1.0; labshift = 12;
```

```
graph4 = phaser[sol4];
```



```
period[sol4]
```

```
6.66
```

We look at the last 10 points in the solution.

```
points = Take[sol4, -10]
```

```
{{13.3, 2.0084, 0.0304849}, {13.32, 2.00859, -0.0102867},
 {13.34, 2.008, -0.0486543}, {13.36, 2.00666, -0.0847496},
 {13.38, 2.00462, -0.118704}, {13.4, 2.00193, -0.150645},
 {13.42, 1.99861, -0.180698}, {13.44, 1.99471, -0.208985},
 {13.46, 1.99026, -0.235623}, {13.48, 1.9853, -0.260724}}
```

We construct initial vectors from the first and last points in the list above.

```
initvec1 = Drop[points[[1]], 1]
```

```
{2.0084, 0.0304849}
```

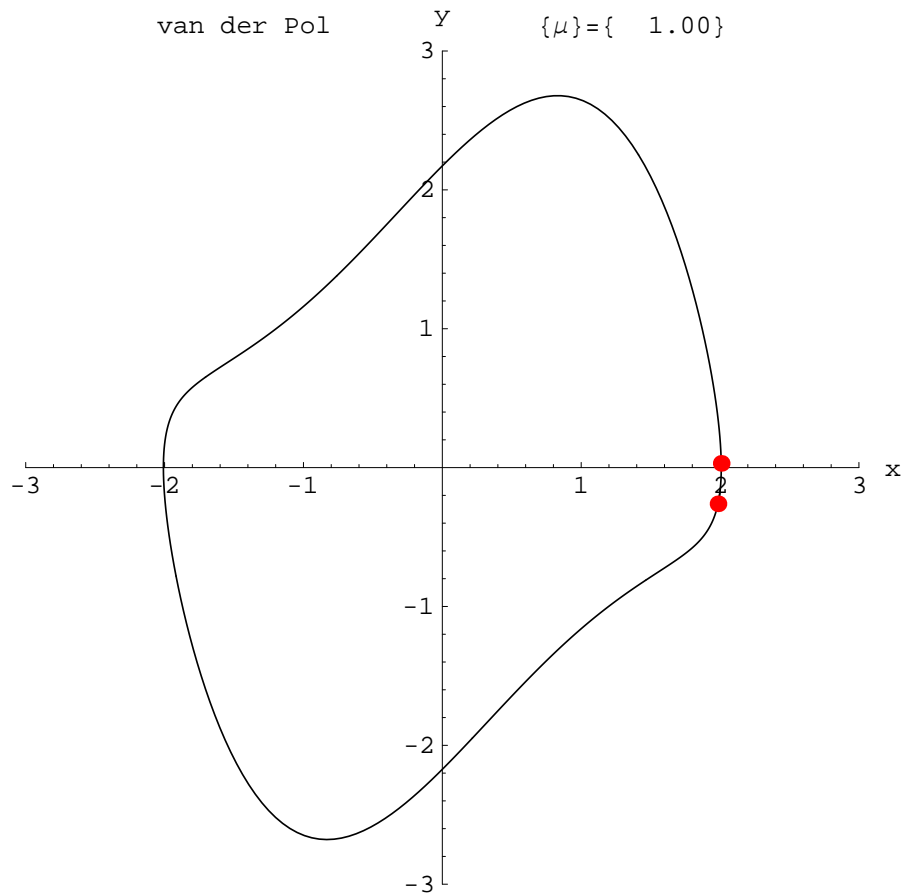
```
initvec2 = Drop[points[[10]], 1]
```

```
{1.9853, -0.260724}
```

Now we construct a phase sequence showing two periods, using the above two initial conditions. Double click on the graph below to start the movie. In the printed version of this notebook only the first frame of the movie is shown.

```
setcolor[{Red, Black, Black}];
```

```
phaseseq[{initvec1, initvec2}, t0, h, 5, 133, 1, 2, graph4];
```



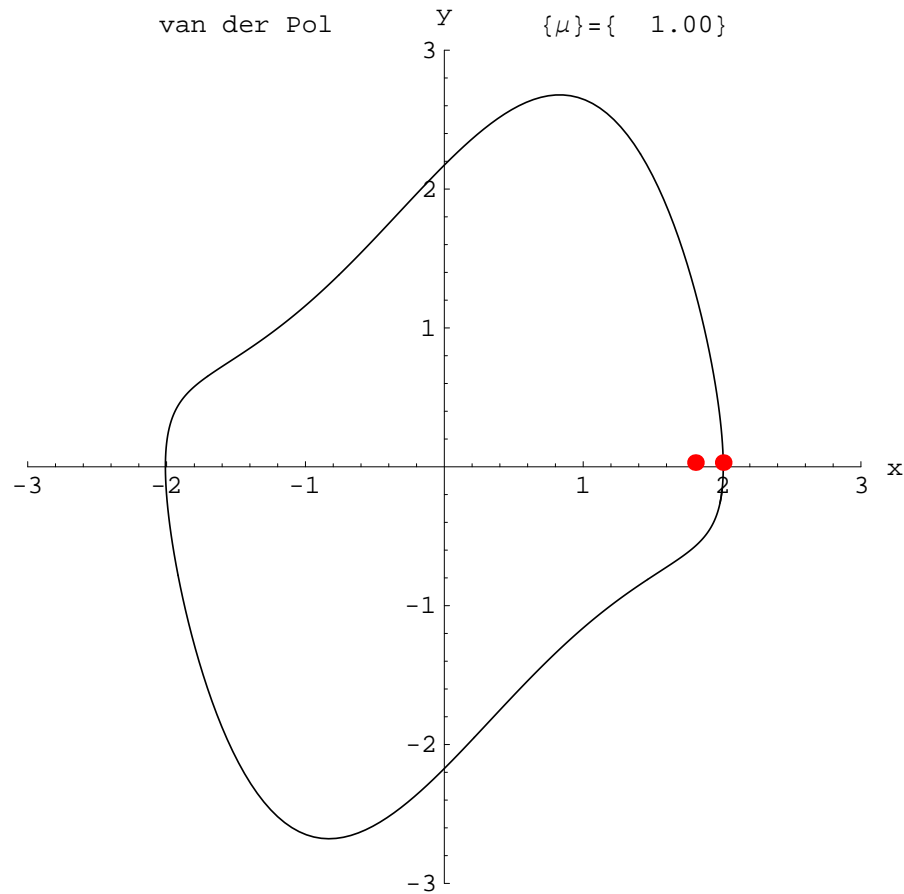
We construct a second movie. This time one initial point is displaced to the interior of the cycle by 0.2 in x. Double click on the graph below to start the movie. In the printed version of this notebook only the first frame of the movie is shown.

```
initvec2 = initvec1 - {0.2, 0}
```

```
{1.8084, 0.0304849}
```

```
setcolor[{Red, Black, Black}];
```

```
phaseseq[{initvec1, initvec2}, t0, h, 5, 133, 1, 2, graph4];
```



We construct a third movie. This time the two initial points, although close, are both off the limit cycle. As the movie will show us, the points remain close together throughout their approach to the cycle and their movement around the cycle, consistent with the stability (but not strict stability) of Van der Pol cycle. Double click on the graph below to start the movie. In the printed version of this notebook only the first frame of the movie is shown.

```
refvec1 = Drop[points[[1]], 1]
```

```
{2.0084, 0.0304849}
```

This is a point on the cycle. We construct initial vectors slightly displaced to either side.

```
initvec1 = refvec1 + {0.075, 0}
```

```
{2.0834, 0.0304849}
```

```
initvec2 = refvec1 - {0.075, 0.0}
```

```
{1.9334, 0.0304849}
```

```
setcolor[{Red, Black, Black}];
```

```
phaseseq[{initvec1, initvec2}, t0, h, 5, 133, 1, 2, graph4];
```

