ME 406
Examples of Stability by Linearization in Nonlinear Systems

```
In[322]:= sysid

        Mathematica 4.1, DynPac 10.65, 2/4/2002
```

■ Introduction

In this notebook we look at a few examples of the determination of stability in nonlinear systems, using the method of linearization. In the first example, linearization gives us a definite conclusion about the stability. In the second and third examples, we consider cases where linearization is inconclusive.

■ Example 1

This is example 10.6 from **Nonlinear Ordinary Differential Equations**, second edition, D.W. Jordan and P. Smith, Oxford Press 1987. The equations are

$$\dot{x} = y(x+1), \quad \dot{y} = x(1 + y^3) .$$

We define this system for DynPac.

```
In[323]:= setstate[{x, y}]; setparm[{}];

In[324]:= slopevec = {y (x + 1), x (1 + y^3)};
```

We find the equilibrium states.

```
In[325]:= eqstates = findpolyeq

Out[325]= {{-1, -1}, {-1, (-1)^(1/3)}, {-1, -(-1)^(2/3)}, {0, 0}}
```

We are interested only in the real ones, which we name eq1 and eq2.

```
In[326]:= eq1 = eqstates[[1]]

Out[326]= {-1, -1}

In[327]:= eq2 = eqstates[[4]]

Out[327]= {0, 0}
```

We ask for the eigenvalues and eigenvectors at each equilibrium.

```
In[328]:= eigsys[eq1]

Out[328]= {{-3, -1}, {{0, 1}, {1, 0}}}
```

Thus eq1 is a strictly stable node, both for the linearized and exact systems.

```
In[329]:= eigsys[eq2]

Out[329]= {{-1, 1}, {{-1, 1}, {1, 1}}}
```

Thus eq2 is a saddle, both for the linearized and exact systems. We also may determine the stability with the DynPac function classify2D.

```
In[330]:= classify2D[eq1]

          Abbreviations used in classify2D.

          L = linear, NL = nonlinear, R2 = repeated root.

          Z1 = one zero root, Z2 = two zero roots.

          This message printed once.


          strictly stable - node

In[331]:= classify2D[eq2]

          unstable - saddle
```

In this example, linearization gave us the answer to the stability question for the two equilibria of the nonlinear system. Next we look at an example for which linearization fails.

■ Example 2

Here we consider two closely related nonlinear systems, each with a single equilibrium point at the origin. These examples also were taken from Jordan and Smith. The first system is

$$\dot{x} = y - x\,(x^2 + y^2)\,,\;\; \dot{y} = -x - y\,(x^2 + y^2)\quad .$$

We define this system for DynPac and call it NL1.

```
In[332]:= setstate[{x, y}]; setparm[{}];

In[333]:= slopevec = {y - x (x^2 + y^2), -x - y (x^2 + y^2)};

In[334]:= sysname = "NL1";
```

It is obvious that the origin is the only equilibrium point. We check the stability, first by the eigenvalues and then equivalently by classify2D.

```
In[335]:= eigval[{0, 0}]
```

```
Out[335]= {-ⅈ, ⅈ}
```

The eigenvalues have zero real part, which means the stability of the equilibrium in the nonlinear system is not determined by this analysis.

```
In[336]:= classify2D[{0, 0}]

        stable (L), indeterminate (NL) - center
```

Classify2D gives us the same answer.

We now save the definition of this system, and then define a second closely related nonlinear system.

```
In[337]:= savesys[NL1];

In[338]:= setstate[{x, y}]; setparm[{}];
```

In[339]:= **slopevec** = $\{y + x\,(x^2 + y^2),\ -x + y\,(x^2 + y^2)\}$;

```
In[340]:= sysname = "NL2";
```

This new system differs from NL1 only in that the nonlinear terms have the opposite sign. Because the linear terms are the same, we expect the same results from the analysis of stability by linearization. Let's check that.

```
In[341]:= eigval[{0, 0}]

Out[341]= {-ⅈ, ⅈ}

In[342]:= classify2D[{0, 0}]

        stable (L), indeterminate (NL) - center
```

Exactly the same. Now we save this system under the name NL2.

```
In[343]:= savesys[NL2];
```

Finally, we define the linearized system associated with both of these systems, the linearization being about the equilibrium at {0,0}.

```
In[344]:= setstate[{x, y}]; setparm[{}];

In[345]:= slopevec = {y, -x};

In[346]:= sysname = "LIN";

In[347]:= savesys[LIN];
```

The relation between these three systems and stability will become very clear if we make a phase portrait of each system. We start with the linearized system. We use portrait and a selected set of initial conditions. We use range checking and integrate forward in time.

```
In[348]:= rangeflag = True; bothdirflag = False;

In[349]:= plrange = {{-1, 1}, {-1, 1}}; ranger = {{-1.1, 1.1}, {-1.1, 1.1}};
```

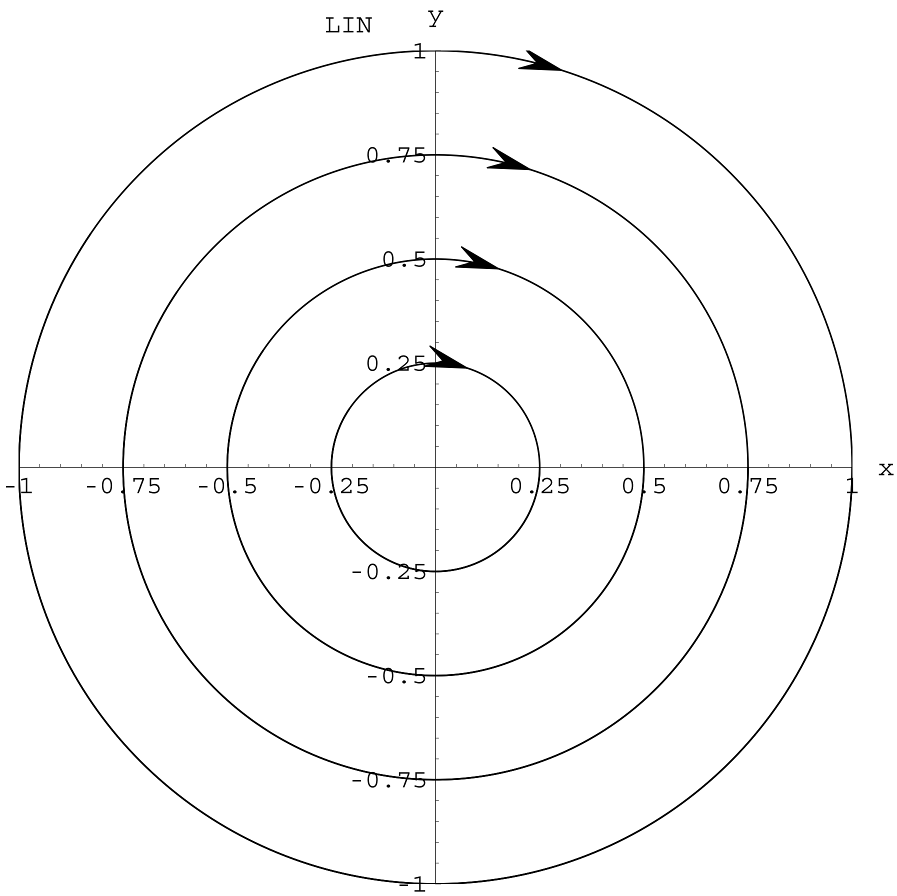
```
In[350]:= t0 = 0.0; h = 0.02; nsteps = 500;
```

```
In[351]:= initvecs = {{1, 0}, {0.75, 0}, {0.5, 0}, {0.25, 0}};
```

```
In[352]:= arrowflag = True; arrowvec = {1 / 2};
```

```
In[353]:= labshift = 10;
```

```
In[354]:= portrait[initvecs, t0, h, nsteps, 1, 2];
```



The equilibrium in the linear system is clearly a center. The orbits neither grow away from the origin nor spiral inward. Thus the system is in a delicate balance. We cannot determine the stability of the equilbrium in the nonlinear system because the linearized system cannot tell us which way the small nonlinear terms will push the orbits. We carry out the integrations for the two nonlinear systems to check this.

First we restore the nonlinear system definition for NL1.

```
In[355]:= restoresys[NL1];
```

We check this system definition by sysreport.

In[356]:= **sysreport**

```
SYSTEM DEFINITION (10.65)


System name sysname = NL1

State vector statevec = {x, y}

State units stateunits = {, }

Slope vector slopevec = {y - x * (x^2 + y^2), -x - y * (x^2 + y^2)}

Parameter vector parmvec = {}

Parameter values parmval = {}

Parameter units vector parmunits = {}

Time unit timeunit =

System Type = differential equation
```

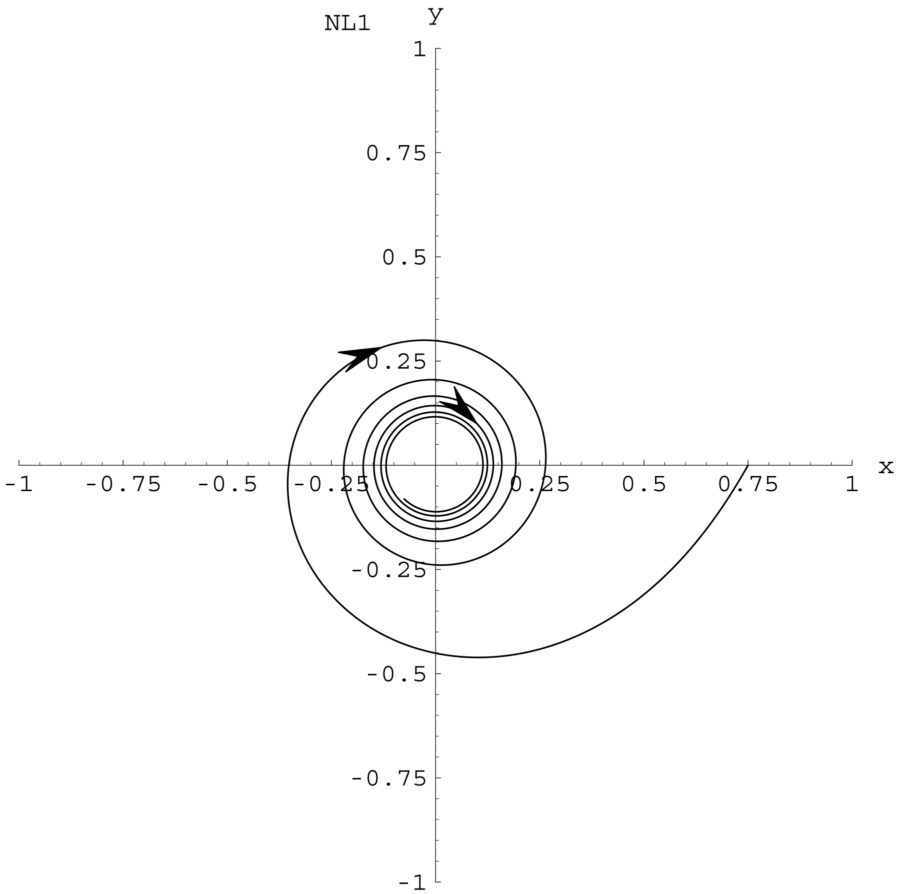
Now we construct the phase portrait. We use a single set of initial conditions and carry out the integration for a long time. Because our analytical work in class showed that this equilibrium in this system is stable, we start out at {0.75,0} and integrate forward in time. We put two arrows on the curve.

In[357]:= **nsteps = 2000;**

In[358]:= **arrowvec = {1 / 4, 3 / 4};**

In[359]:= **initvecs = {{0.75, 0}};**

```
In[360]:= portrait[initvecs, t0, h, nsteps, 1, 2];
```



The curve is spiralling into the origin, suggesting stability. This is consistent with the analytical solution found in class which showed strict stability for this equilibrium. Because the nonlinear terms get very small as we approach the equilibrium, the spiral is quite slow.

Now we look at NL2, which differs from NL1 only in the algebraic sign of the nonlinear terms.

```
In[361]:= restoresys[NL2];
```

```
In[362]:= sysreport

SYSTEM DEFINITION (10.65)


    System name sysname = NL2

    State vector statevec = {x, y}

    State units stateunits = {, }

    Slope vector slopevec = {y + x * (x^2 + y^2), -x + y * (x^2 + y^2)}

    Parameter vector parmvec = {}

    Parameter values parmval = {}

    Parameter units vector parmunits = {}

    Time unit timeunit =

    System Type = differential equation
```

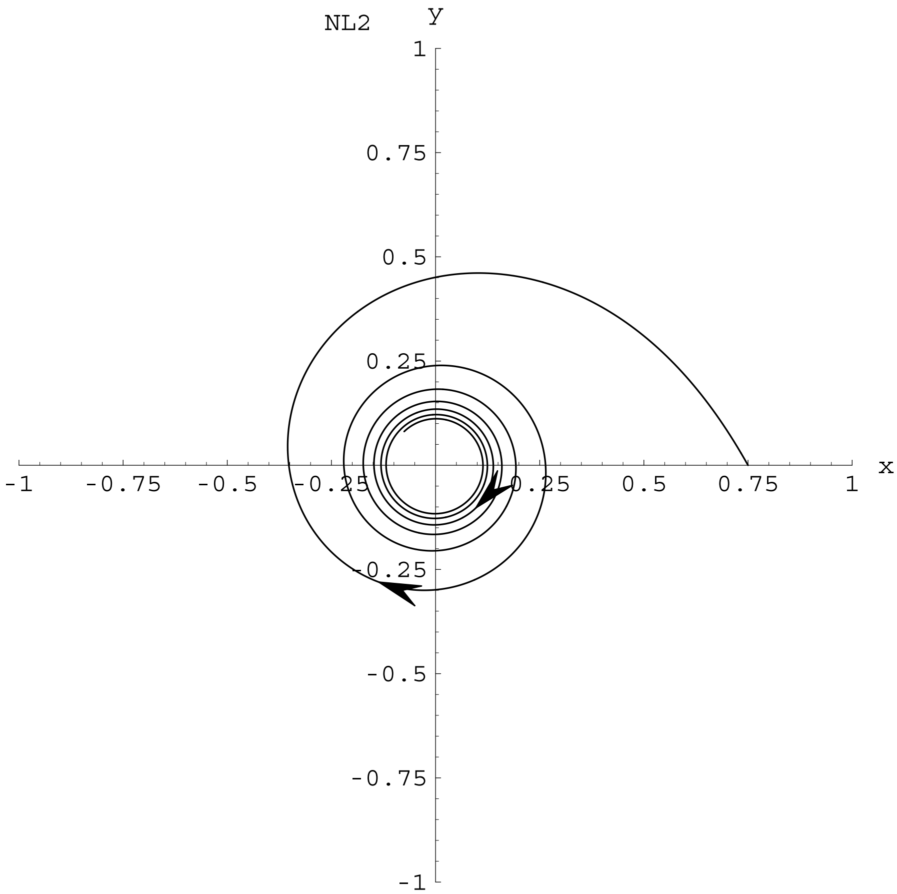
We construct the phase portrait. We suspect (from our work in class) that this equilibrium is unstable, so we start near the outer edge of our window and integrate backwards in time.

```
In[363]:= h = -0.02; nsteps = 2000;

In[364]:= arrowvec = {1 / 4, 3 / 4};

In[365]:= initvecs = {{0.75, 0}};
```

```
In[366]:= portrait[initvecs, t0, h, nsteps, 1, 2];
```



These curves spiral outward, suggesting strongly the instability established in class by the analytical solution.

■ Example 3

The next example is what we call a strongly nonlinear system about the equilibrium at the origin. (There may be other equilibria for this system. We do not consider them here.) The system is

$$\dot{x} = a_1\, x^2 + 2\, b_1\, x\, y + c_1\, y^2\,,\;\; \dot{y} = a_2\, x^2 + 2\, b_2\, x\, y + c_2\, y^2\,.$$

If we linearize this system about $x = 0$ and $y = 0$, the equations reduce to a non-informative $\dot{x} = 0$ and $\dot{y} = 0$. More formally: the **A** matrix for the linearized system has all zero entries and two zero eigenvalues. Clearly in this case, we get no conclusion about the stability of the equilibrium.

■ Conclusions

Stability of an equilibrium is a local concept. The stability is determined by the behavior of the system in the immediate vicinity of the equilibrium. This suggests that one may analyze the stability by linearizing the system about the equilibrium point. Most of the time (Example 1 above) this works. If the linearized system itself is either strictly stable or unstable, restoring the nonlinear terms, which are very small near equilibrium, will not change the qualitative behavior of the system in its choice of going toward or away from the equilibrium. In some cases, however (Example 2 above), the linearized system is exactly balanced, and the orbits of the linearized system enclose the equilbrium without either going toward it or away from it. In those cases, linearization gives no conclusion about stability. The balance is tipped either way by the small nonlinear terms which are ignored in the linearization analysis. In still other cases (Example 3 above), the nonlinearity is very strong, and linearization throws away essential information about the system. For Examples 2 and 3, we need more refined techniques, such as the Liapunov method to be discussed in class.