ME 406
Example of Stable and Unstable Manifolds

```
intreset; plotreset;
```

```
sysid
```

```
Mathematica 4.1.2, DynPac 10.66, 3/8/2002
```

Introduction

In this notebook, we study an example given in **Differential Equations and Dynamical Systems**, Lawrence Perko, second edition, Springer-Verlag, 1996, p. 111. The equations are

$$\dot{x} = -x - y^2, \quad \dot{y} = x^2 + y \ . \tag{1}$$

This system has an equilibrium point at (0,0). We will construct both the linear and nonlinear stable and unstable manifolds in the vicinity of the origin. Then we will construct the global stable and unstable manifolds.

System Definition

We define the system for DynPac.

```
setstate[{x,y}];
```

```
setparm[{}];
```

```
slopevec = {-x - y^2, y + x^2};
```

```
parmval = {};
```

```
sysname = "nlsystem";
```

```
intreset;
```

```
plotreset;
```

Equilibrium Point

We start by analyzing the equilibrium at the origin. We first use classify2D, and then we use eigers to obtain the eigenvectors and eigenvalues at the origin.

```
classify2D[{0,0}]
```

```
Abbreviations used in classify2D.

L = linear, NL = nonlinear, R2 = repeated root.

Z1 = one zero root, Z2 = two zero roots.

This message printed once.


unstable - saddle
```

```
eigers = eigsys[{0, 0}]
```

```
{{-1, 1}, {{1, 0}, {0, 1}}}
```

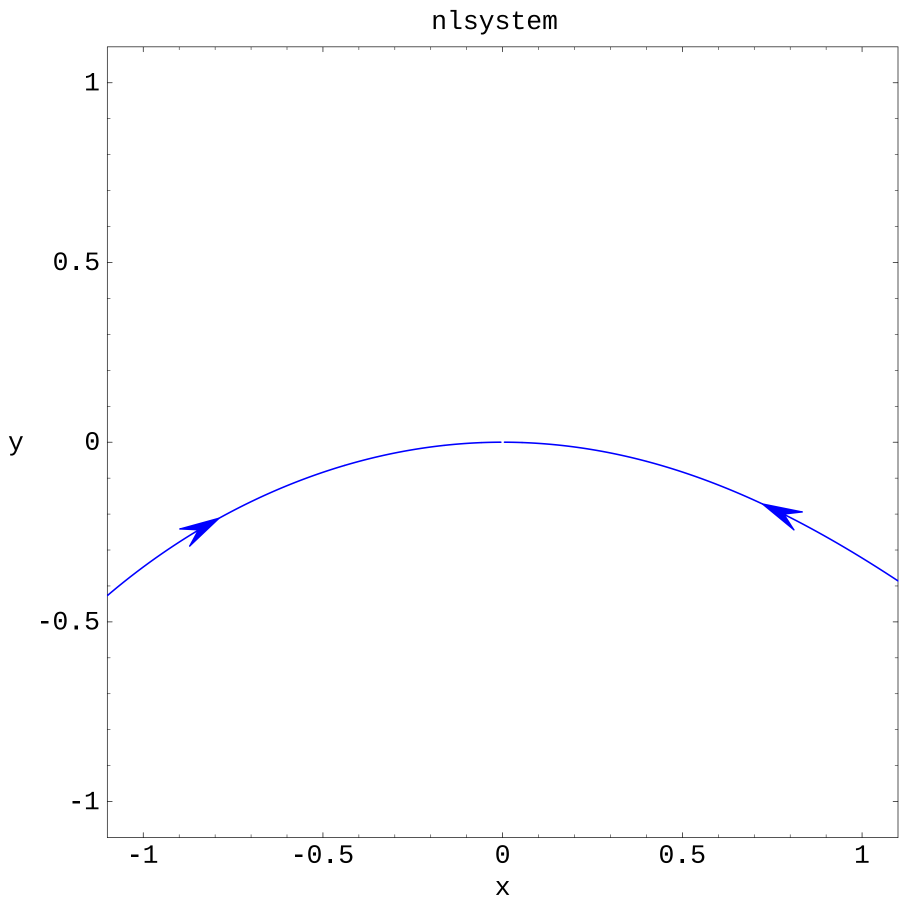
Stable and Unstable Manifolds for Nonlinear System

We will use blue for stable manifolds, red for unstable manifolds.

The stable manifold corresponds to the negative eigenvalue. To construct it, we integrate backwards in time, starting very near the equilibrium, displaced from the equilibrium along the eigenvector associated with the eigenvalue -1. The solution going backwards in time could grow very rapidly, so we avoid the possibility of overflow and crash by using range checking. We limit the range to a box from -1 to 1 in both variables. We add an arrow to the mid-point of each orbit. We remove the axes (and put a frame around the picture) so that the tangencies can be better seen when we compare linear and nonlinear systems.

```
ranger = {{-1.4,1.4},{-1.4,1.4}};

rangeflag = True;

t0 = 0.0;

h = -0.02;

nsteps = 1000;

initset = {0.005*{1,0},-0.005*{1,0}};

arrowflag = True;

arrowvec = {1 / 2};

asprat = 1.0; plrange = {{-1.1, 1.1}, {-1.1, 1.1}};
 axon = False; frameon = True;

setcolor[{Blue}];
```

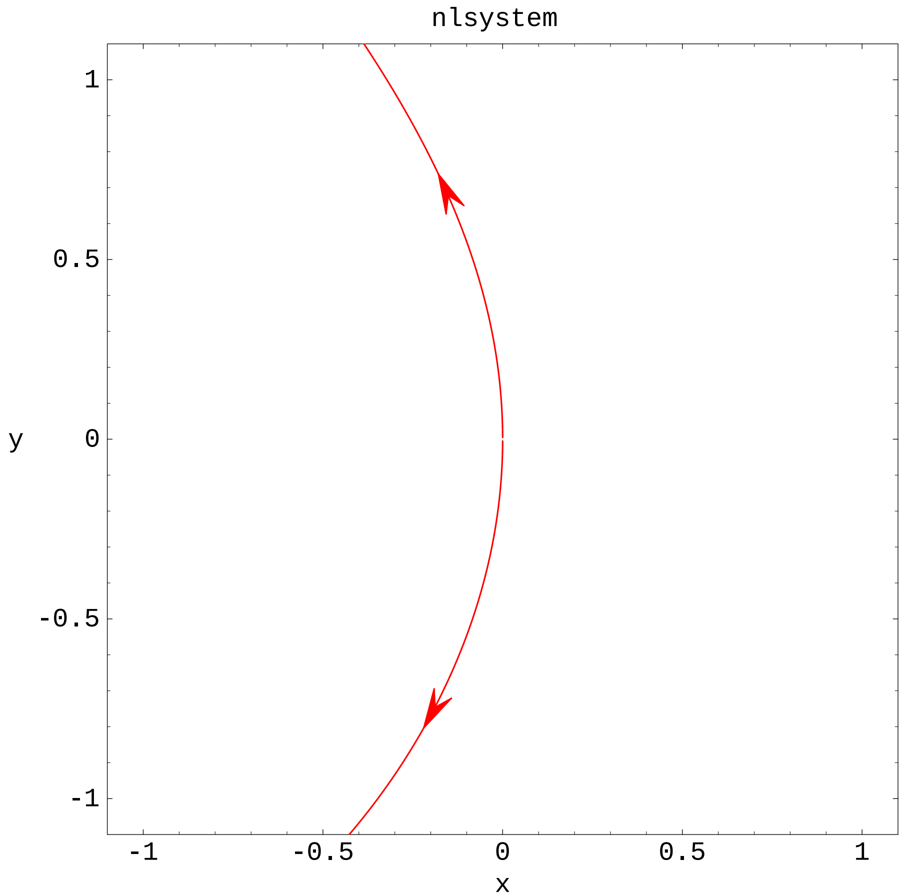
```
stableport = portrait[initset,t0,h,nsteps,1,2];
```



Now we construct the unstable manifold.

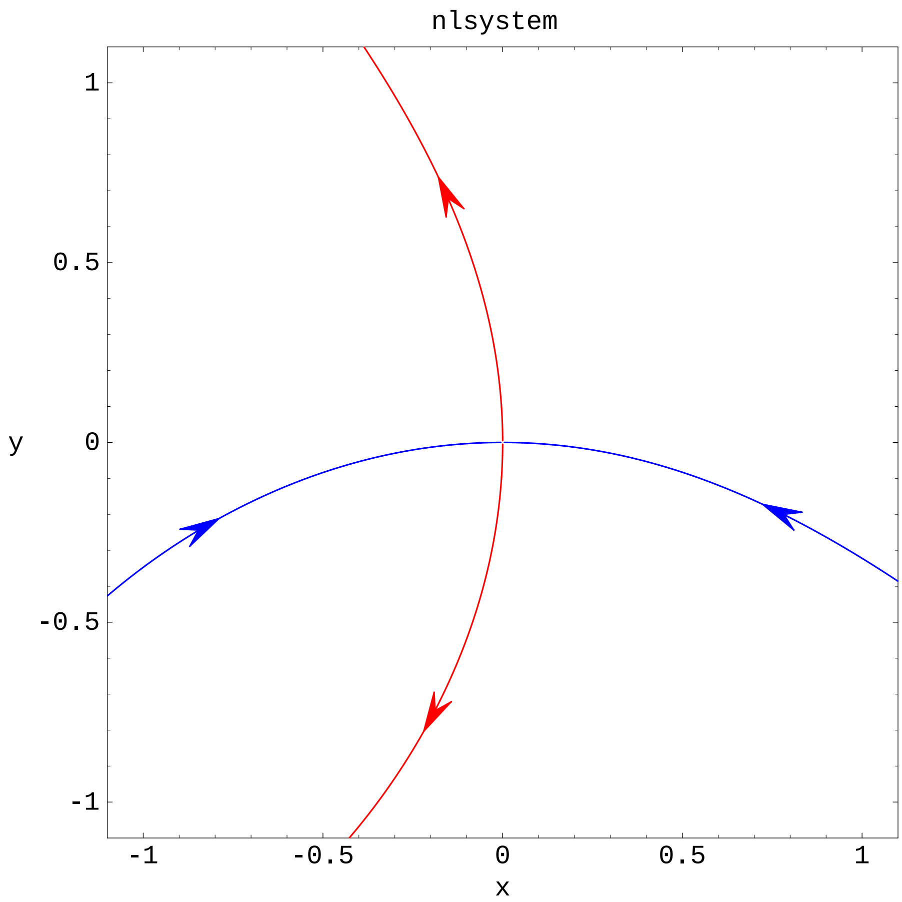
```
setcolor[{Red}];

initset = {0.005*{0,1},-0.005*{0,1}};

h = 0.02;
```

```
unstableport = portrait[initset,t0,h,nsteps,1,2];
```



Now we combine these.

```
nlman = show[stableport, unstableport];
```



■ Stable and Unstable Manifolds for Linearized System

Now we construct the linearized system and then go through the same manifold constructions for it. We first save the nonlinear system under the name nlsystem.

```
savesys[nlsystem];
```

We continue to use {x,y} for the state vector, and both parmval and parmvec remain empty lists. Thus for the linear system, we must redefine only the slope function. It is the derivative matrix at equilibrium times the state vector.

```
slopevec = Dot[dermatval[{0,0}],{x,y}]
```

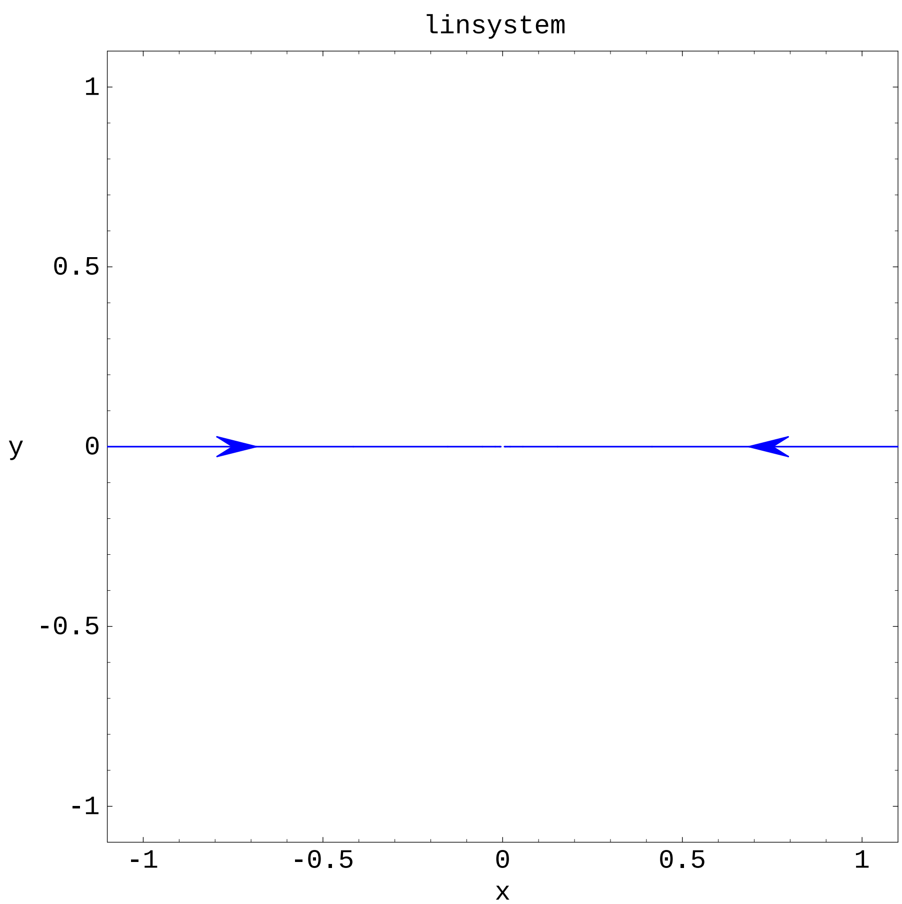
```
{-x, y}
```

```
sysname = "linsystem";
```

Now we construct the linear stable manifold.

```
ranger = {{-1.4,1.4},{-1.4,1.4}};
```

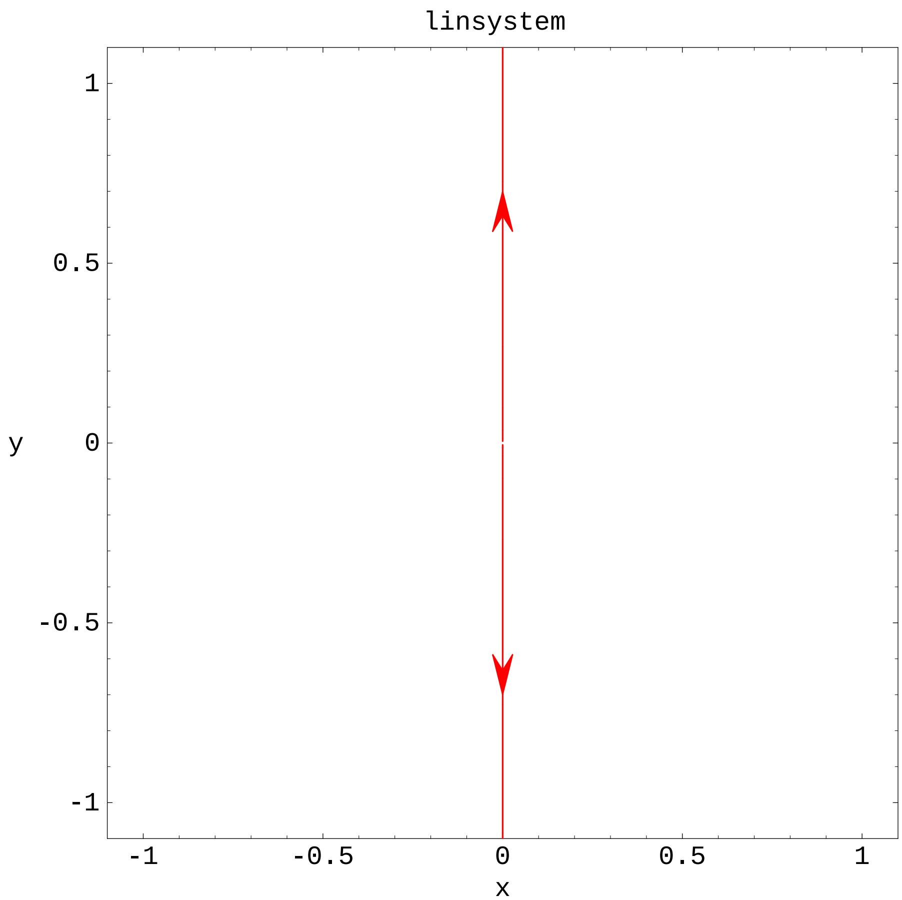
```
rangeflag = True;

t0 = 0.0;

h = -0.02;

nsteps = 1000;

initset = {0.005*{1,0},-0.005*{1,0}};

arrowflag = True;

arrowvec = {1 / 2};

asprat = 1.0; plrange = {{-1.1, 1.1}, {-1.1, 1.1}};
 axon = False;

setcolor[{Blue}];

linstableport = portrait[initset,t0,h,nsteps,1,2];
```



Now we construct the unstable manifold.

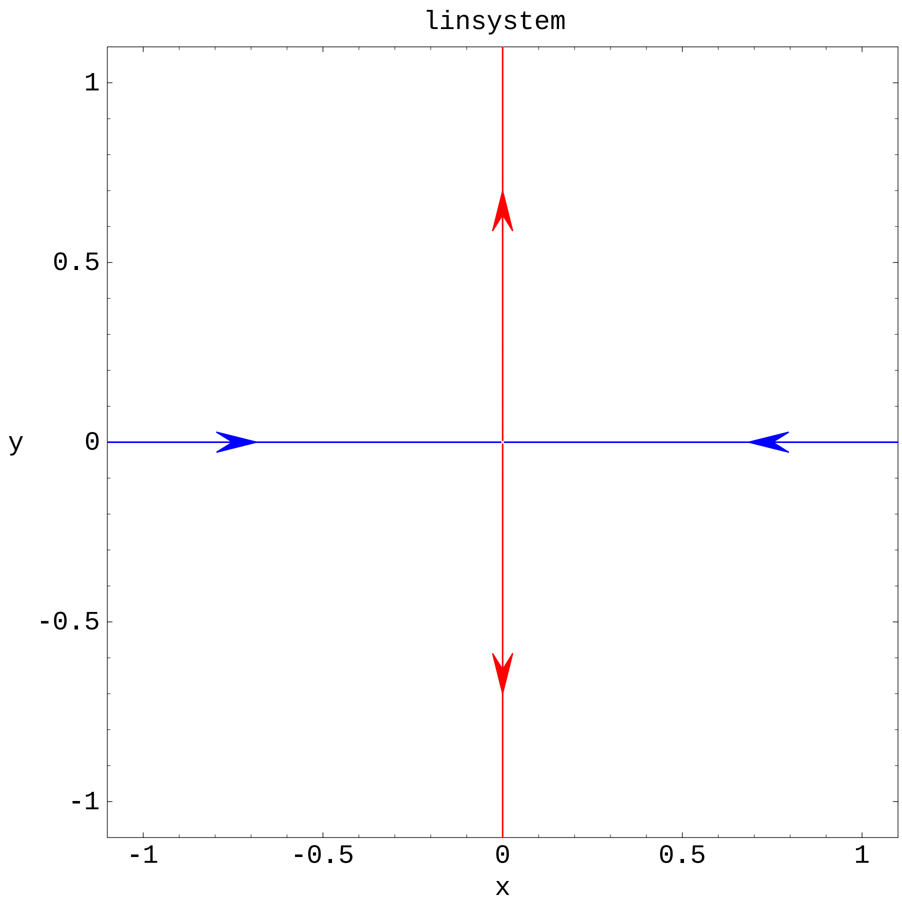
```
setcolor[{Red}];

initset = {0.005*{0,1},-0.005*{0,1}};
```

```
h = 0.02;
linunstableport = portrait[initset,t0,h,nsteps,1,2];
```



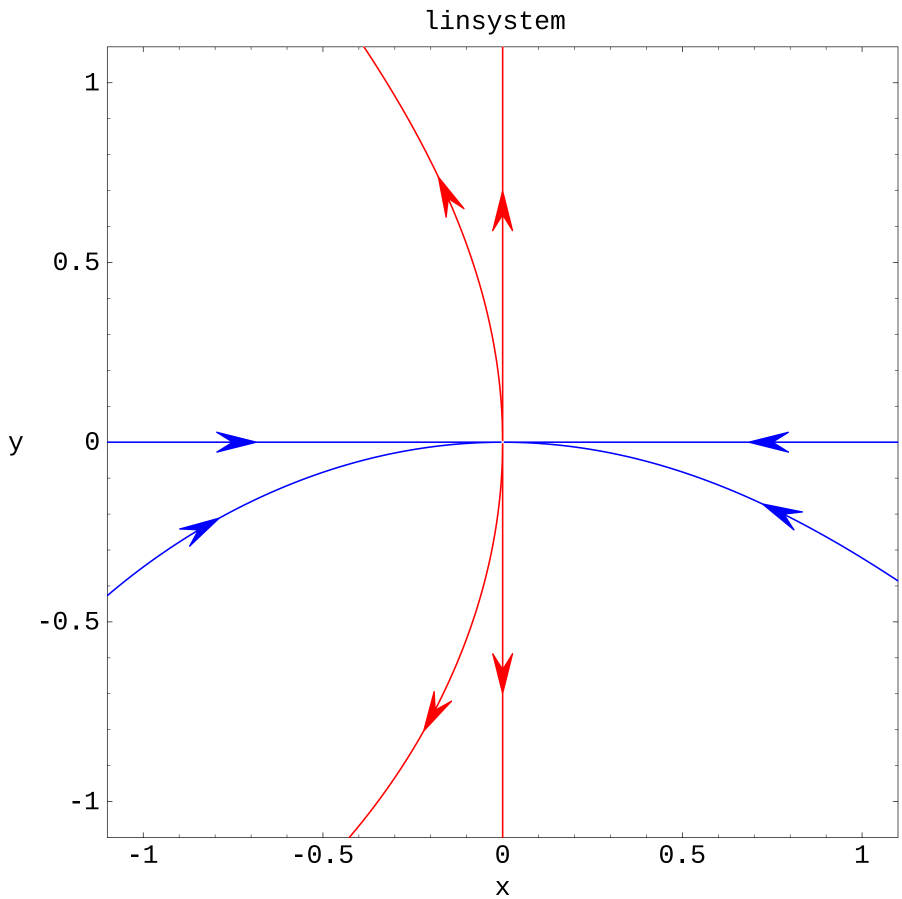
We combine the last two pictures.

```
lman = show[linstableport, linunstableport];
```



Now for the finale, showing both the linear and nonlinear manifolds and illustrating the tangencies at the equilibrium point.

```
show[lman, nlman];
```



■ Global Manifolds for Nonlinear System

Now we restore the nonlinear system, and we try to find the global stable and unstable manifolds. We will get some surprises in that effort.

```
restoresys[nlsystem];
```

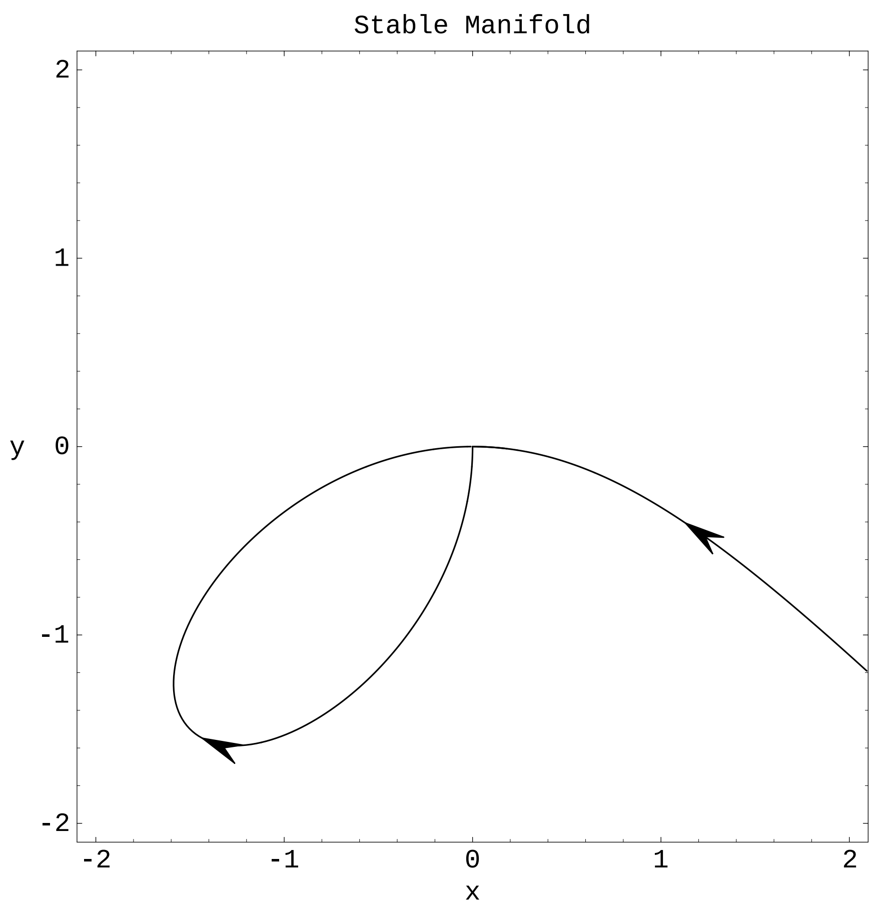
We enlarge our plotting window, and we begin with the construction of the stable manifold. We start on the eigenvectors very near the origin and integrate backwards in time.

```
eps = 0.01;

initset = {eps * {1, 0}, eps * {-1, 0}};

t0 = 0.0; h = -0.02; nsteps = 1000;

plrange = {{-2.1, 2.1}, {-2.1, 2.1}}; asprat = 1.0;

rangeflag = True; ranger = plrange; bothdirflag = False;

setcolor[{Black}];
```

```
labon = "Stable Manifold";
```

```
arrowflag = True; arrowvec = {1 / 2};
```

```
globestab = portrait[initset,t0,h,nsteps,1,2];
```



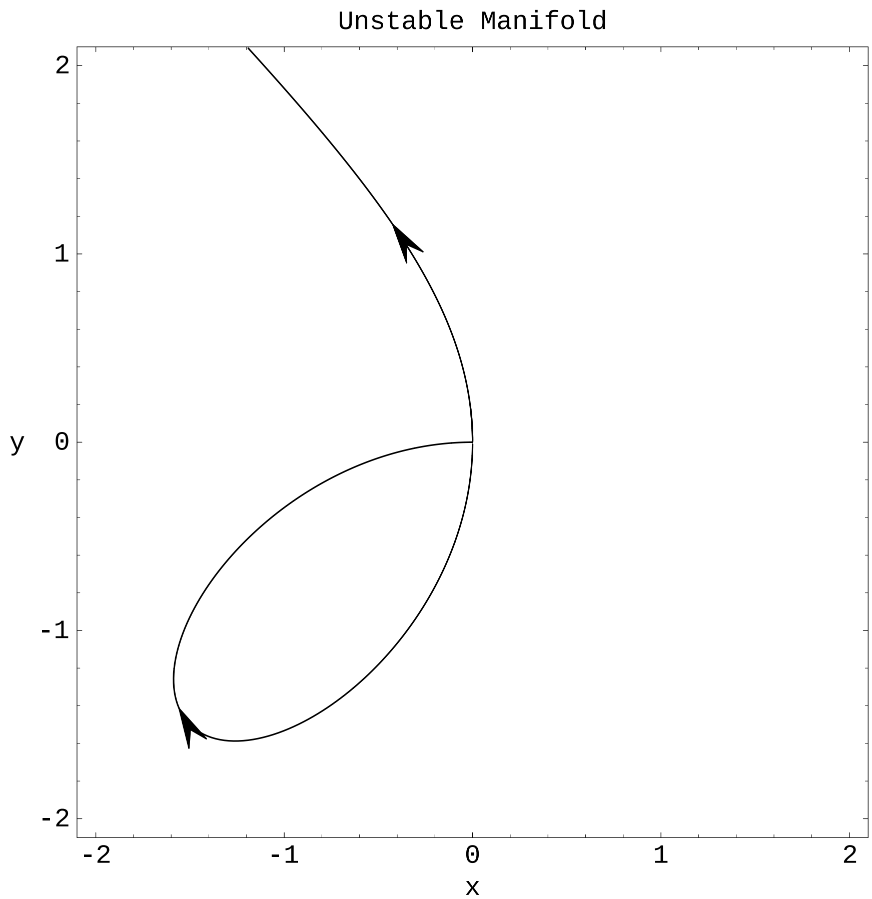
Now we carry this out for the unstable manifold. For this, we start near the equilibrium on the eigenvectors and integrate forward in time.

```
initset = {eps * {0, 1}, eps * {0, -1}};
```

```
h = 0.02;
```

```
labon = "Unstable Manifold";
```

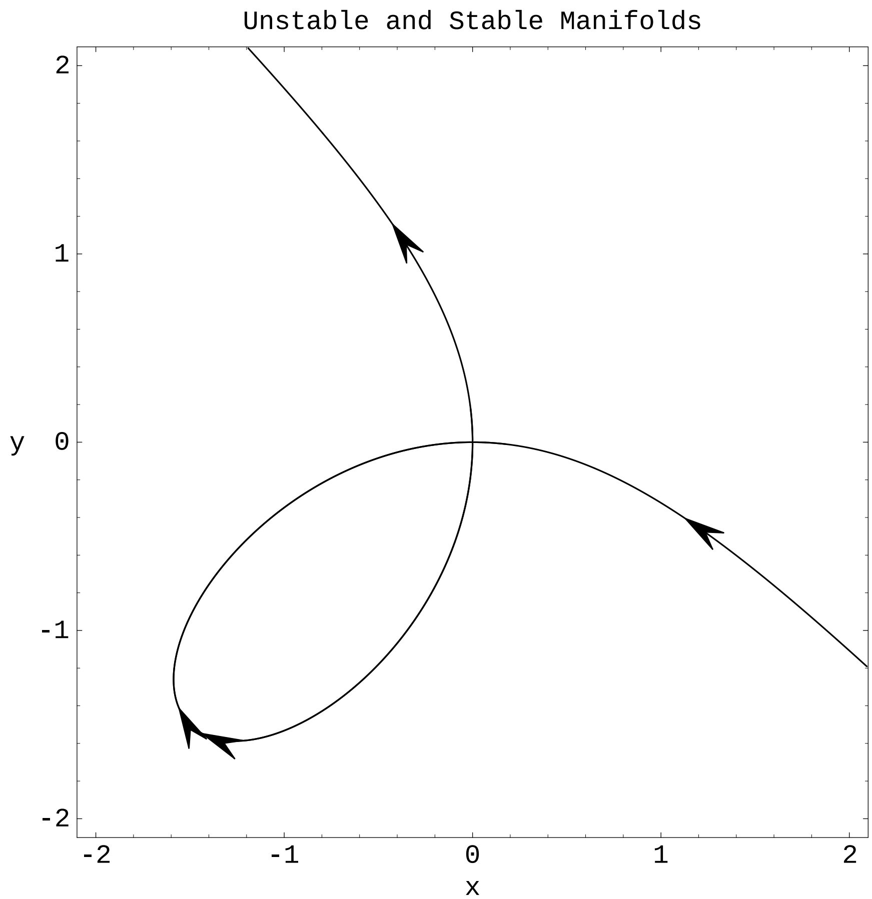
```
globeunstab = portrait[initset,t0,h,nsteps,1,2];
```



The biggest surprise here is the homoclinic loop rooted in the origin: it is part of both the global unstable and global stable manifold! A little thought shows that this doesn't violate the rules, whether or not we imagined such a thing could happen. If we start on the loop and go forward in time, we arrive at the origin, hence it is part of the stable manifold. If we start on the loop and go backward in time, we also arrive at the origin, so it is part of the unstable manifold. We can show the two manifolds together:

```
labon = "Unstable and Stable Manifolds";
```

```
show[globestab, globeunstab];
```



Although the homoclinic loop is not periodic, it is pretty clear that there is a periodic solution or solutions in its interior, and also an equilibrium point. The equilibrium point is easy to find -- it is {-1,-1}, also confirmed by DynPac:

```
findpolyeq
```

$\{\{-1, -1\}, \{0, 0\}, \{(-1)^{1/3}, -(-1)^{2/3}\}, \{-(-1)^{2/3}, (-1)^{1/3}\}\}$

We classify the new equilibrium.

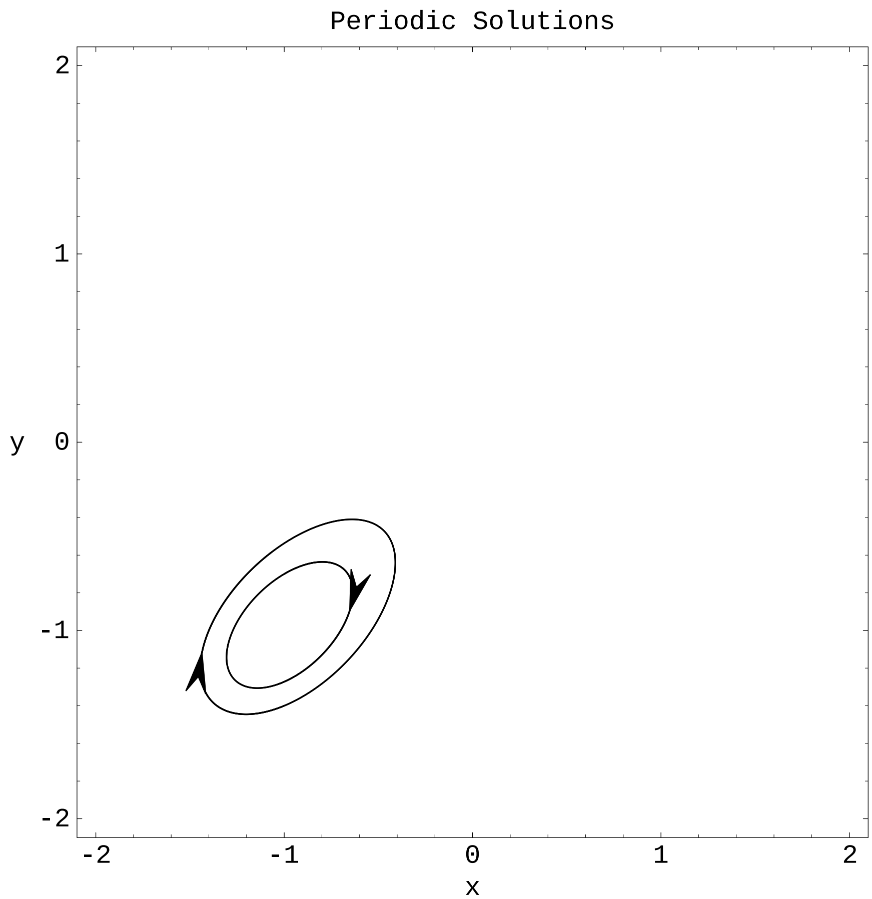
```
classify2D[{-1, -1}]
```

stable (L), indeterminate (NL) - center

Let's try a few initial conditions inside the loop.

```
initset = {{-1, -1.4}, {-1, -0.7}};

t0 = 0.0; h = 0.02; nsteps = 500;

arrowvec = {1 / 2};

labon = "Periodic Solutions";
```

```
periodic = portrait[initset,t0,h,nsteps,1,2];
```

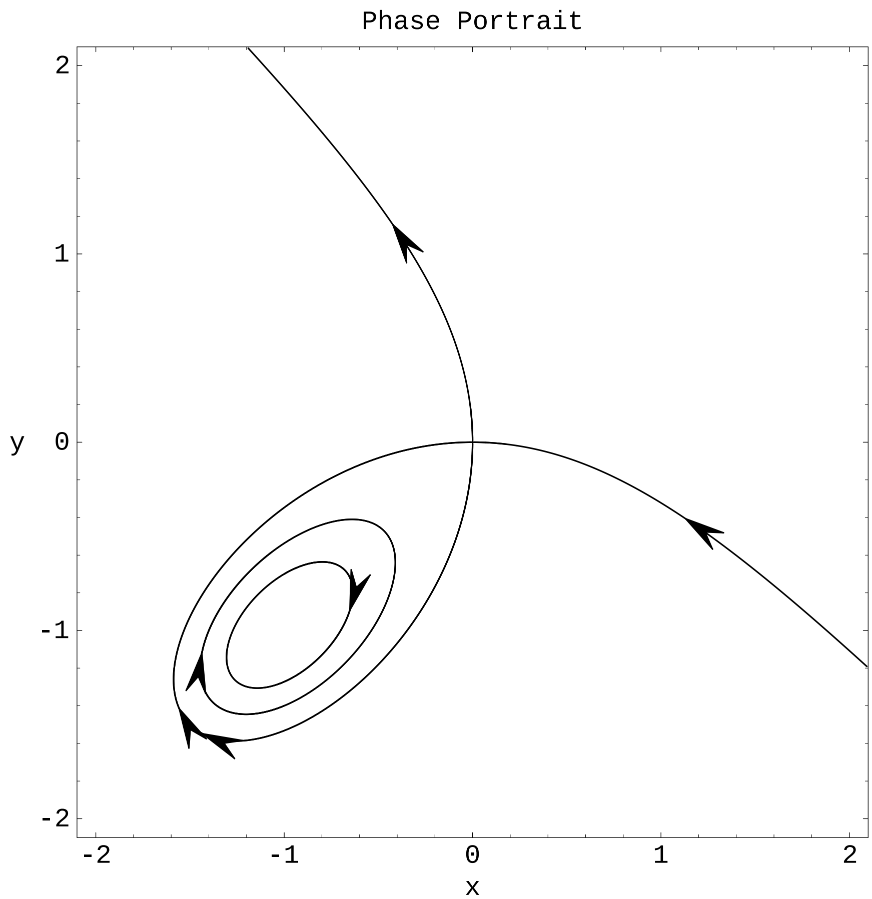


Thus the equilibrium at {-1,-1} is a nonlinear center.

We put all of this together.

```
labon = "Phase Portrait";
```

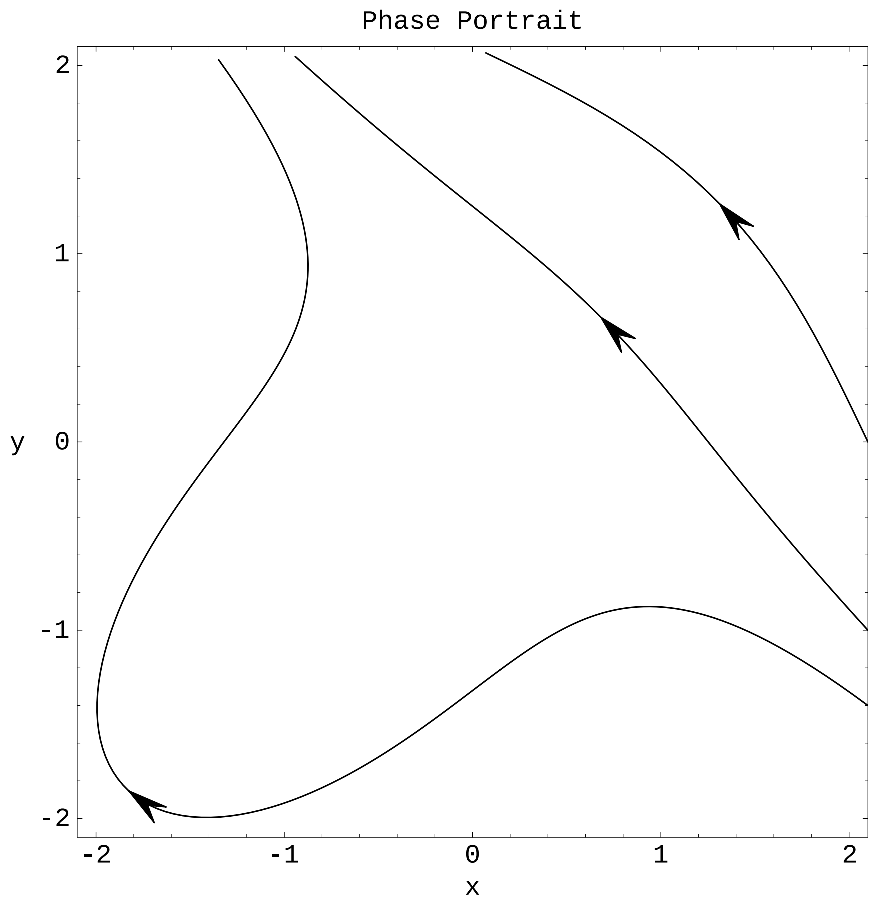
```
show[globestab, globeunstab, periodic];
```



We add two more orbits for completeness. Two in the upper right, and a third which curves around the outside of the homoclinic loop.

```
initset = {{2.1, 0}, {2.1, -1.4}, {2.1, -1}};
```

```
moreport = portrait[initset,t0,h,nsteps,1,2];
```



```
show[globestab, globeunstab, periodic, moreport];
```