ME 406
Logistic Model for US Population

■ Fitting the Logistic Model to the US Population in 1921

The year is 1921. We are going to fit the logistic population model to the US census data from the time period 1790 - 1920. We hope from the fit to make a prediction of the maximum sustainable US population. The basic differential equation is for the logistic model is

$$\frac{dP}{dt} = k\,P(1 - P/PM)\ .$$

As shown in class, the solution to the initial value problem is

```
In[1]:= plog[t_, PM_] := PM / (1 + ((PM - P0) / P0) * Exp[-k * (t - t0)])
```

Here PM is the maximum sustainable population, P0 is the initial population, k is the growth rate, and t0 is the initial time. Before using this function, we will need to supply values for these four constants.

We use the US census data from the initial census in 1790 up to the census of last year (1920). This data is available in a number of different differential equations texts, and also on the US Census web site. We store the data in a list named census.

```
In[34]:= census = {{1790, 3.93}, {1800, 5.31}, {1810, 7.24}, {1820, 9.64},
           {1830, 12.87}, {1840, 17.07}, {1850, 23.19}, {1860, 31.44}, {1870, 39.82},
           {1880, 50.16}, {1890, 62.95}, {1900, 75.99}, {1910, 91.97}, {1920, 105.71}};
```

We can retrieve any element of the list by using the index of the element. For example, to get the third pair, we type

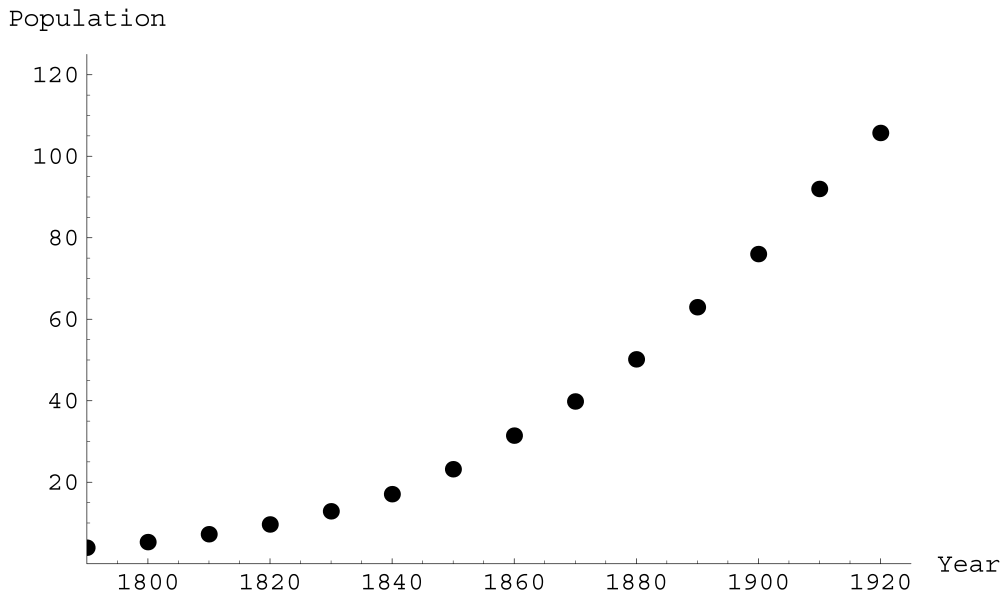
```
In[3]:= census[[3]]

Out[3]= {1810, 7.24}
```

The final time in the data table we assign to the variable tf.

```
In[36]:= tf = 1920;
```

We begin our analysis by plotting this data. We call the graph censusgraph. The relevant plotting command is ListPlot, which plots a list of coordinate pairs. We use an option to make the dots bigger and hence a little easier to see. We ask that each dot have a width equal to 0.02 times the width of the plot.

```
In[16]:= censusgraph =
          ListPlot[census, AxesLabel -> {"Year", "Population"}, AxesOrigin -> {1790, 0},
           PlotRange -> {{1790, tf + 5}, {0, 125}}, PlotStyle -> PointSize[0.02], ImageSize -> 380];
```



To use the logistic model for predictive purposes, we must use the data to determine the values of the parameters in the model. The initial time and the initial population are gotten directly from the data:

```
In[6]:= t0 = 1790; P0 = 3.93;
```

As shown in class, we can estimate the growth rate k from the data for early years, in which the population is far below P_M and hence the growth is Malthusian. We somewhat arbitrarily use the data points for 1790 and 1810 to estimate k. The estimate comes from the Malthusian law applied to the growth from the initial time 1790 to 1810

$$P(t) = P_0 \, Exp[k*(t - t0)] \quad ,$$

from which we get

$$k = Log[P(t)/P(t0)]/(t - t0) \quad .$$

We use *Mathematica* to calculate k, using the data from 1790 and 1810:

In[7]:= $k = \frac{1}{1810 - 1790} * Log[7.24 / 3.93]$

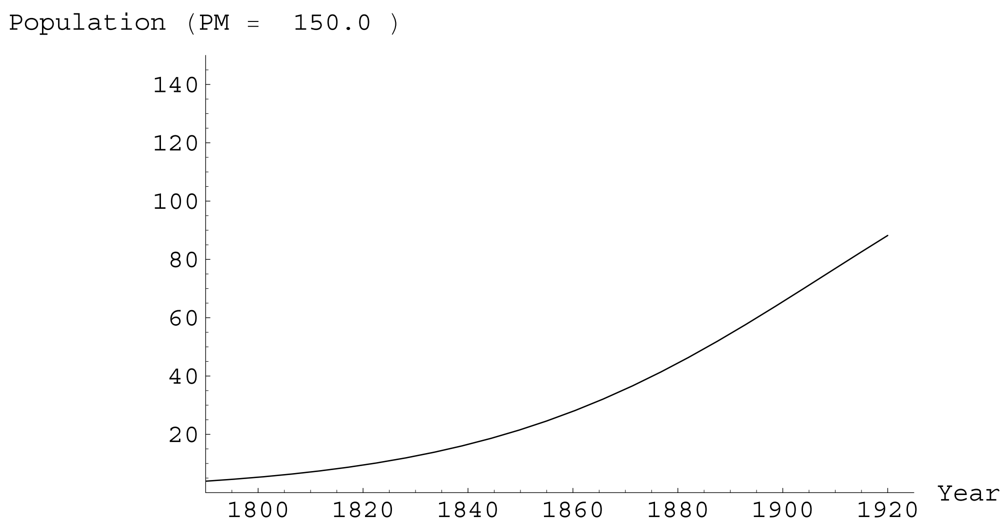
Out[7]= 0.0305491

The remaining parameter to determine is the maximum sustainable population PM. Although we could use a sophisticated numerical procedure to do this, it is more instructive to use a graphical trial-and-error procedure. Basically we will pick a value of PM and then plot the theoretical curve and the actual data on a single graph. We will vary PM until we get a match, if possible. We define a routine now called logistigraph which when executed will construct the theoretical graph with a supplied value of PM, and an upper plotting value of PTOP

```
In[32]:= logisticgraph[PM_, PTOP_] := Plot[plog[t, PM], {t, t0, tf}, AxesLabel ->
           {"Year", SequenceForm["Population (PM = ", PaddedForm[PM, {4, 1}], " )" ]},
          AxesOrigin -> {1790, 0}, PlotRange -> {{1790, tf + 5}, {0, PTOP}}, ImageSize -> 380];
```

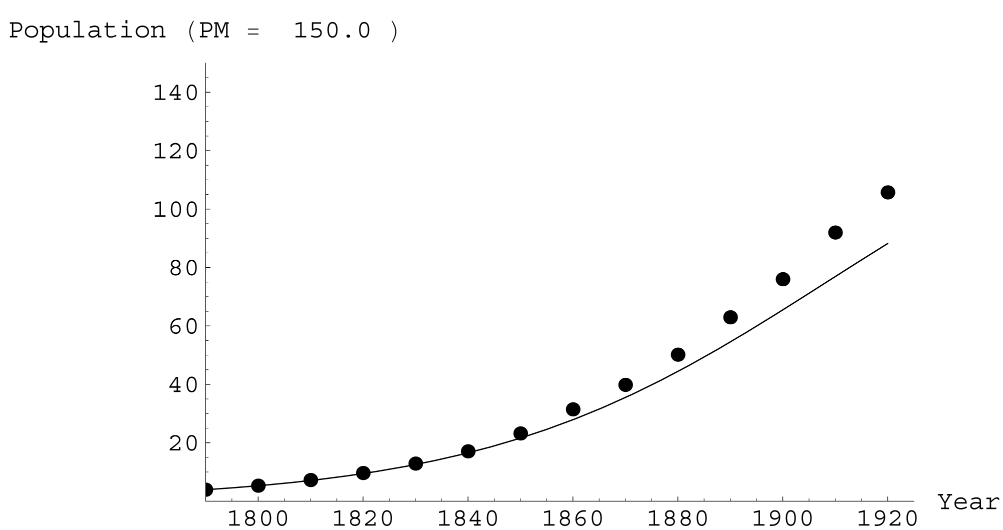
Let's try this out for PM = 150.

```
In[37]:= graph150 = logisticgraph[150, 150];
```



Now we use *Mathematica*'s show command to combine this with the data:

```
In[38]:= Show[graph150, censusgraph];
```



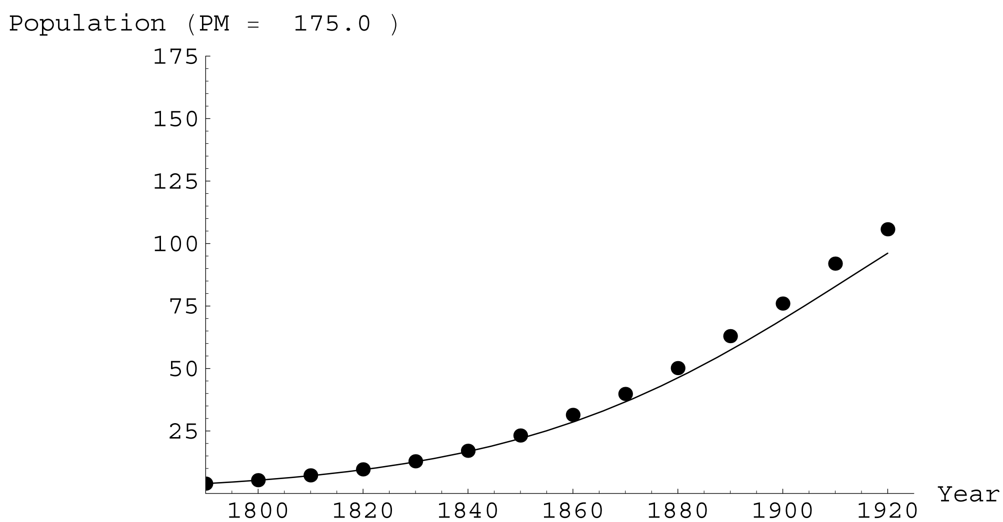
We see that the fit is not too bad, but the theoretical curve lies below the data. We will try larger values of PM. First we automate the process by defining now a routine which will construct the theoretical graph and combine it with the data graph all in one operation. We call this routine compare. We first redefine logisticgraph so that it does not display separately.

```
In[39]:= logisticgraph2[PM_, PTOP_] := Plot[plog[t, PM], {t, t0, tf}, AxesLabel ->
            {"Year", SequenceForm["Population (PM = ", PaddedForm[PM, {4, 1}], " )" ]},
           AxesOrigin -> {1790, 0}, PlotRange -> {{1790, tf + 5}, {0, PTOP}},
           DisplayFunction -> Identity, ImageSize -> 380];
```

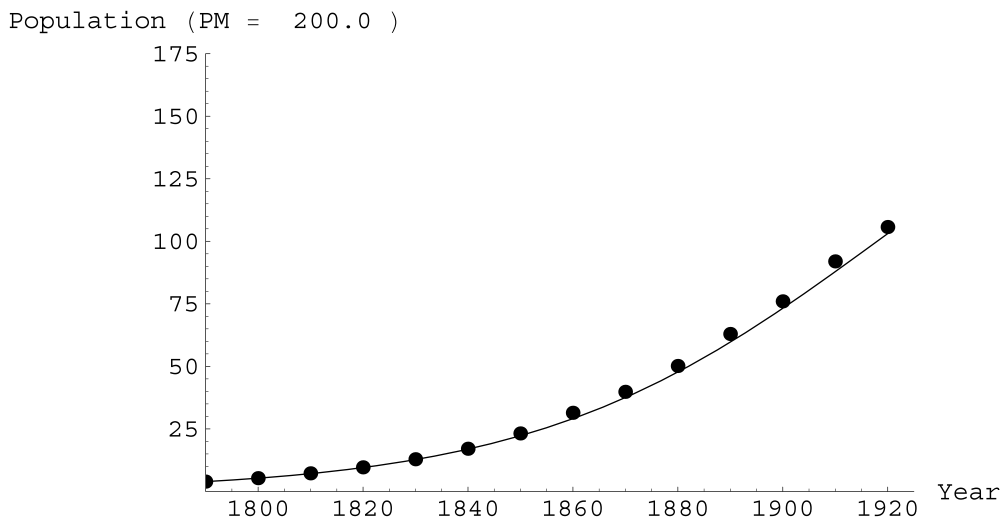
```
In[40]:= compare[PM_, PTOP_] :=
          Show[logisticgraph2[PM, PTOP], censusgraph, DisplayFunction -> $DisplayFunction]
```

We will now try this for a sequence of increasing PM values.

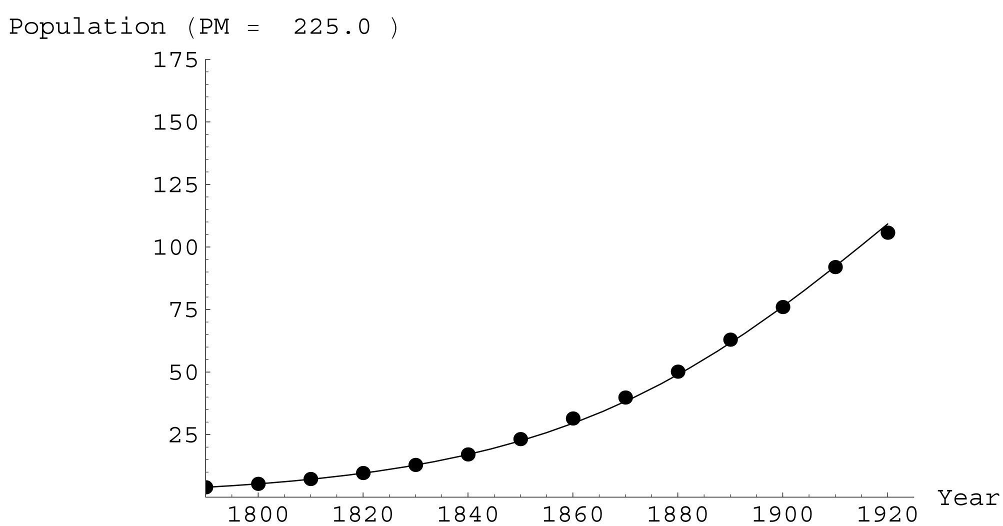
```
In[41]:= compare[175, 175];
```



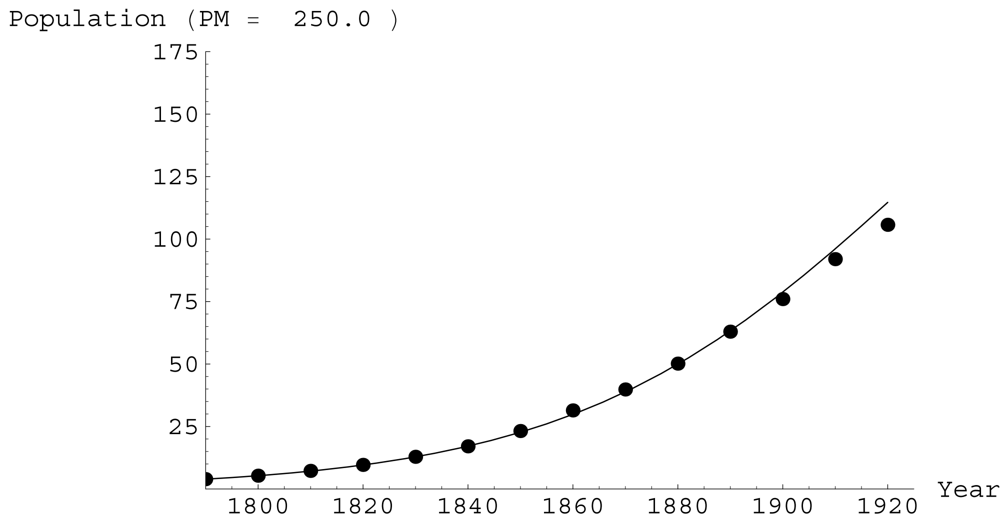
```
In[42]:= compare[200, 175]
```




```
Out[42]= - Graphics -

In[43]:= compare[225, 175];
```



```
In[44]:= compare[250, 175];
```



Tthe best fit appears to be for PM = 225.

■ What the 1921 Model Says in 2002

The fit for PM = 225 appears to be excellent. Thus in 1921, we could have achieved a certain amount of fame by claiming that our mathematical model predicts a maximum US population of 225 million. As the years go by, however, our fame will turn to notoriety, if we are even remembered at all. Let's see why that is so by fast-forwarding to 2002. Now our data list is

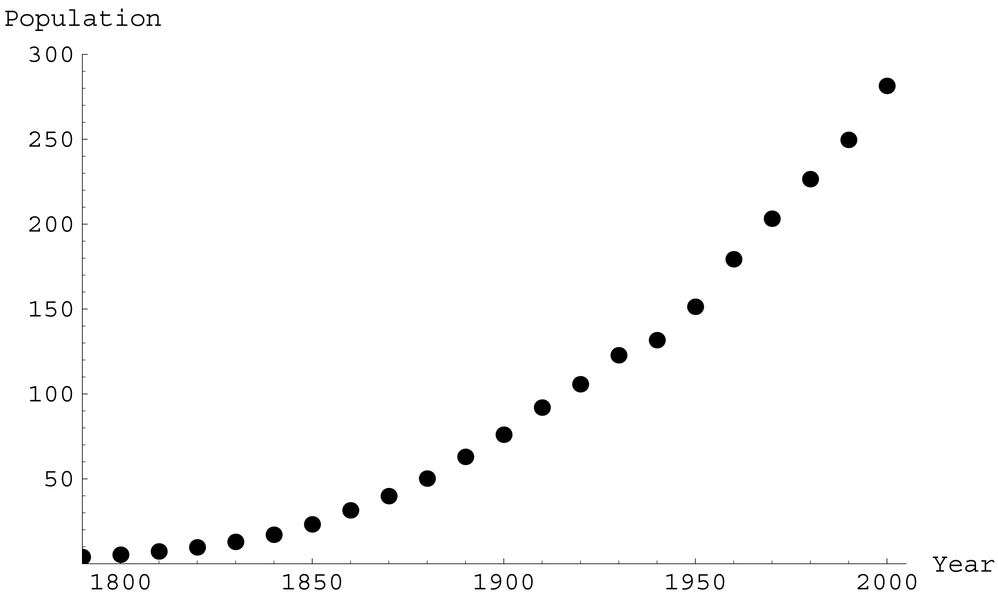
```
In[45]:= census = {{1790, 3.93}, {1800, 5.31}, {1810, 7.24}, {1820, 9.64},
            {1830, 12.87}, {1840, 17.07}, {1850, 23.19}, {1860, 31.44}, {1870, 39.82},
            {1880, 50.16}, {1890, 62.95}, {1900, 75.99}, {1910, 91.97}, {1920, 105.71},
            {1930, 122.78}, {1940, 131.67}, {1950, 151.33}, {1960, 179.32},
            {1970, 203.21}, {1980, 226.50}, {1990, 249.63}, {2000, 281.42}};
```

We change our final time to

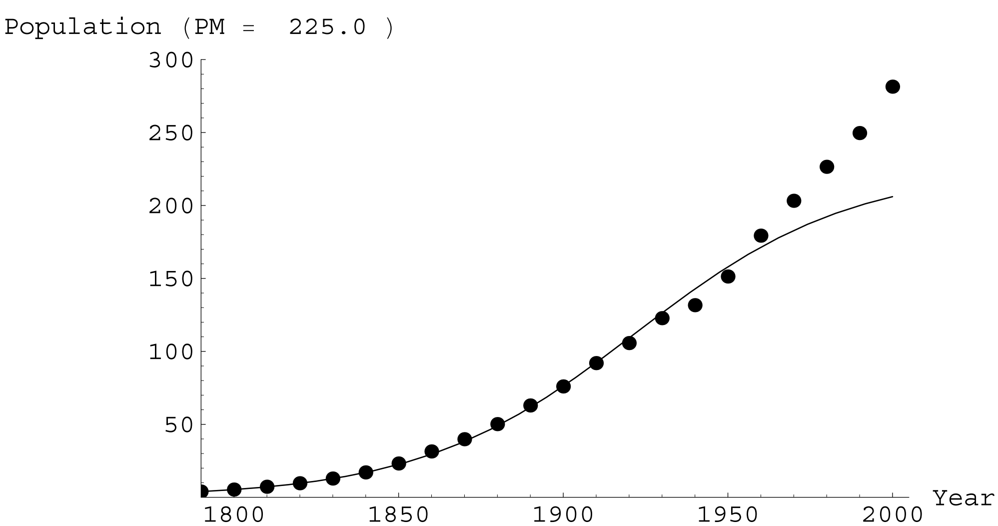
```
In[46]:= tf = 2000;
```

and replot both the data and the model for PM = 250.

```
In[47]:= censusgraph =
          ListPlot[census, AxesLabel -> {"Year", "Population"}, AxesOrigin -> {1790, 0},
           PlotRange -> {{1790, tf + 5}, {0, 300}}, PlotStyle -> PointSize[0.02], ImageSize -> 380];
```




```
In[48]:= compare[225, 300];
```



It is clear from the graph that this is not a good fit. The data are showing a very different trend than the model. We also know that the present population of the US is well over the predicted theoretical maximum of 225 million. The point is that the logistic model with only two fixed parameters (k and PM) should not be expected to predict the population in a complex, time-dependent and evolving culture. Of course the question of the maximum sustainable population in the US or other countries or the entire world is a crucial one, but one on which experts do not agree, and it is a question that is not going to be answered by a simple model with two parameters.

■ Who Was Thomas Malthus?

In 1798, the English clergyman, professor, and political economist Thomas R. Malthus published anonymously a pamphlet entitled **An Essay on the Principle of Population as It Affects the Future Improvement of Society, with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers**. (A number of inexpensive modern reprints are available -- for example, the one edited by Geoffrey Gilbert and published by Oxford University Press in 1993.) This pamphlet was meant to counter some of the rosier utopian views of that time on the perfectability of human society. Malthus' counter argument was that population growth would eventually produce widespread poverty or worse. In his words (p. 61 of the edition cited above):

"Must it not then be acknowledged by an attentive examiner of the histories of mankind, that in every age and in every state in which man has existed, or does now exist,

That the increase of population is necessarily limited by the means of subsistence.

That population does invariably increase when the means of subsistence increase. And,

That the superior power of population is repressed, and the actual population kept equal to the means of subsistence, by misery and vice."

These were not popular words in 1798 nor are they now. The debate started by Malthus continues to this day.

Thomas R. Malthus
(1766 - 1834)