

# ME 406

## Bifurcations VII

### Subcritical Hopf Bifurcation

```
sysid
```

```
Mathematica 4.1.2, DynPac 10.66, 3/5/2002
```

```
intreset; plotreset;
```

## 1. Introduction

In this notebook, the seventh in a series of notebooks on bifurcations, we look at a simple example of a subcritical Hopf bifurcation. At the bifurcation, an unstable limit cycle is absorbed by a stable spiral equilibrium. The equilibrium becomes unstable, and the system state will then jump from the unstable equilibrium to some distant attractor, a possibly catastrophic transition. We construct a movie showing the changes in the equilibrium and the limit cycles as the bifurcation parameter changes.

## 2. Definition of the System

We consider the following system, depending on one parameter  $a$ :

$$\dot{x} = -y + x(x^2 + y^2 + \mu)(1 - x^2 - y^2), \quad \dot{y} = x + y(x^2 + y^2 + \mu)(1 - x^2 - y^2)$$

We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}];  
slopevec =  
  {-y + x * (x^2 + y^2 + μ) * (1 - x^2 - y^2), x + y * (x^2 + y^2 + μ) * (1 - x^2 - y^2)};  
sysname = "SubCritical Hopf";
```

We look for equilibrium states.

```
findpolyeq
```

```
{{0, 0}}
```

It is easy to show that this is the only equilibrium. For this problem, it is useful to obtain the slope functions in polar coordinates. We use the function `polartrans` to do this.

```
polarslope = polartrans[x, y, r, θ]
```

```
{-r (-1 + r^2) (r^2 + μ), 1}
```

We have gotten lucky. Almost everything we might want to know about this system can be read from the radial slope function. We see that there is a limit cycle at  $r = 1$ , and, for negative  $\mu$ , another at  $r = \sqrt{-\mu}$ . It is easy to check the stability, by looking at the sign of polarslope as we move slightly away from the limit cycle. We get the following results: for  $\mu < -1$ , the cycle at  $\sqrt{-\mu}$  is stable, and the cycle at 1 is unstable. For  $-1 < \mu < 0$ , the cycle at  $\sqrt{-\mu}$  is unstable, and the cycle at 1 is stable. Thus we see that as  $\mu$  passes through -1, we have a transcritical bifurcation of limit cycles. As  $\mu \rightarrow 0^-$ , the unstable cycle is absorbed by the equilibrium. For  $0 < \mu$ , we have only the stable cycle at 1. Let's check the stability of the equilibrium point.

```
eigsys[{0, 0}]
```

```
{{-i + μ, i + μ}, {{-i, 1}, {i, 1}}}
```

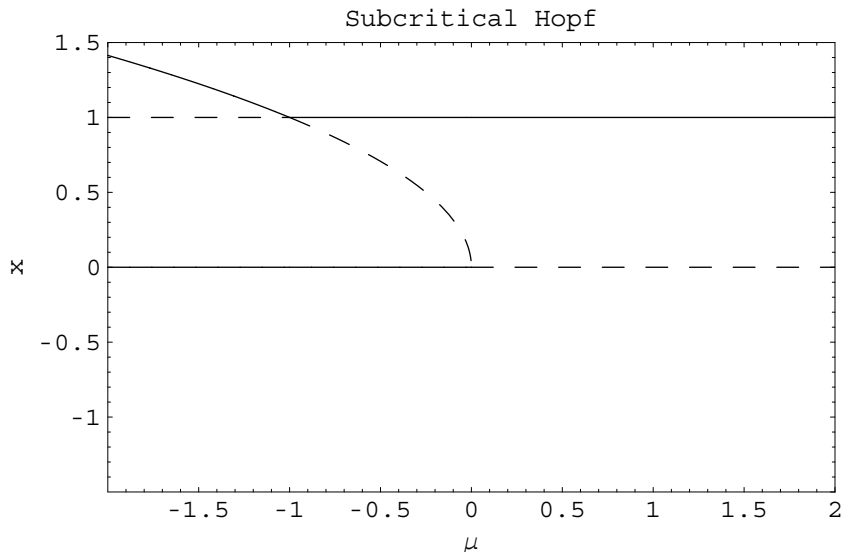
Thus the equilibrium is a stable spiral for  $\mu < 0$ , an unstable spiral for  $\mu > 0$ , and a center in linearized theory for  $\mu = 0$ . It is not hard to show that the equilibrium is unstable for  $\mu = 0$ . The bifurcation at  $\mu = 0$  is called a subcritical Hopf bifurcation. Imagine a system in the stable equilibrium state for  $\mu < 0$ . As we increase  $\mu$ , the unstable limit cycle shrinks to the equilibrium. When  $\mu$  reaches zero, the equilibrium becomes unstable, and the system will make a finite jump to the nearest attractor, namely the limit cycle at  $r = 1$ . Such a large and sudden jump could be catastrophic. A less catastrophic sequence occurs if we start on the stable limit cycle for  $\mu < -1$ . Then as we pass through  $\mu = -1$ , we switch to the stable cycle at  $r = 1$ , and remain there as  $\mu$  continues to increase. We may visualize this in a bifurcation diagram by plotting the position of the equilibria and the amplitude of the limit cycles, using as usual solid for stable, dashed for unstable.

```
plot1 = Plot[{0, Sqrt[-μ], 1}, {μ, -2, -1},
  PlotRange → {{-2, 2}, {-1.5, 1.5}},
  PlotLabel → "Subcritical Hopf",
  FrameLabel → {"μ", "x"}, Axes → False, ImageSize → imsize,
  Frame → True, DisplayFunction → Identity,
  PlotStyle → {Dashing[{0.03, 0.0}], Dashing[{0.03, 0.0}],
  Dashing[{0.03, 0.03}]}];
```

```
plot2 = Plot[{0, Sqrt[-μ], 1}, {μ, -1, 0},
  PlotRange → {{-2, 2}, {-1.5, 1.5}},
  PlotLabel → "Subcritical Hopf", Frame → True,
  FrameLabel → {"μ", "x"}, Axes → False,
  DisplayFunction → Identity, ImageSize → imsize,
  PlotStyle → {Dashing[{0.03, 0.00}], Dashing[{0.03, 0.03}],
  Dashing[{0.03, 0.0}]}];
```

```
plot3 = Plot[{0, 1}, {μ, 0, 2}, PlotRange → {{-2, 2}, {-1.5, 1.5}},
  PlotLabel → "Subcritical Hopf", Frame → True,
  FrameLabel → {"μ", "x"}, Axes → False,
  DisplayFunction → Identity,
  PlotStyle → {Dashing[{0.03, 0.03}], Dashing[{0.03, 0.00}]}];
```

```
Show[{plot1, plot2, plot3}, DisplayFunction -> $DisplayFunction];
```



In previous bifurcation sequences, we have constructed a family of orbits for each  $\mu$ , and then tracked the changes with  $\mu$ . Because there is more going on here with the two limit cycles and the equilibrium, we first make a movie showing only the limit cycles and equilibrium. This shows the essential features of the bifurcation. Then we will make a second movie in which we add two orbits near the equilibrium point. The refgraph we define here has the limit cycles and equilibrium as a function of  $\mu$ .

```
refgraph1 := Module[{ans1, ans2, ans3, rad, u},
  ptsize = 0.03; rad = Sqrt[-First[parmval]]; display = False;
  setcolor[{Blue}]; ans1 = dots[{{0, 0}}]; lnthick = 0.007;
  ans2 = plotcurve[{{rad * Cos[u], rad * Sin[u]}, {u, 0, 2 Pi}}];
  setcolor[{Red}];
  ans3 = plotcurve[{{Cos[u], Sin[u]}, {u, 0, 2 Pi}}];
  lnthick = 0.002; {ans1, ans2, ans3}]

refgraph2 := Module[{ans1, ans2, ans3, rad, u},
  ptsize = 0.03; rad = Sqrt[-First[parmval]]; display = False;
  setcolor[{Blue}]; ans1 = dots[{{0, 0}}]; lnthick = 0.007;
  ans2 = plotcurve[{{Cos[u], Sin[u]}, {u, 0, 2 Pi}}];
  setcolor[{Red}];
  ans3 = plotcurve[{{rad * Cos[u], rad * Sin[u]}, {u, 0, 2 Pi}}];
  lnthick = 0.002; {ans1, ans2, ans3}]

refgraph3 := Module[{ans1, ans2, u}, ptsize = 0.03;
  display = False;
  setcolor[{Red}]; ans1 = dots[{{0, 0}}];
  setcolor[{Blue}]; lnthick = 0.007;
  ans2 = plotcurve[{{Cos[u], Sin[u]}, {u, 0, 2 Pi}}];
  lnthick = 0.002; {ans1, ans2}]
```

```

refgraph := Module[{ans, mu}, mu = First[parmval];
  plrange = {{-1.5, 1.5}, {-1.5, 1.5}}; asprat = 1;
  labshift = 15;
  If[(mu < -1), (ans = refgraph1),
    (If[(mu < 0), (ans = refgraph2),
      (ans = refgraph3) ])]];
  setcolor[{Black}]; display = True; ans]

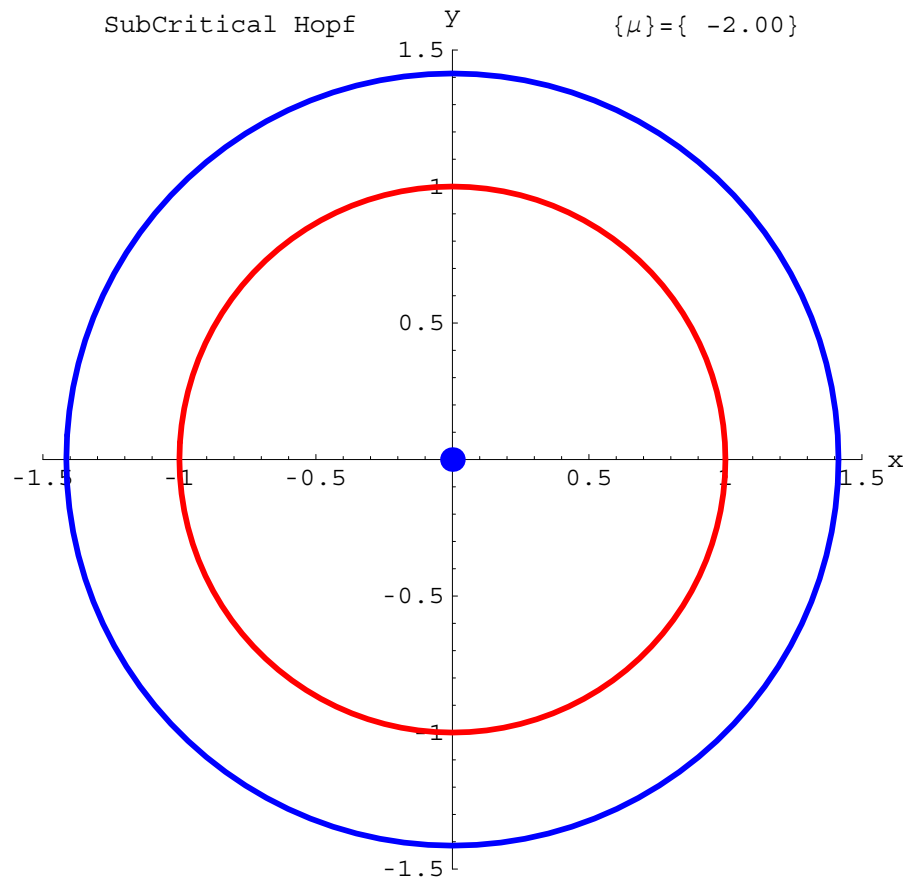
```

Now we use a Do loop to construct the movie with just the limit cycles and the equilibrium. As usual we use blue for stable and red for unstable.

```

Do[parmval = {i * 0.05}; show[refgraph], {i, -40, 20}];

```



Now we repeat the graph construction, this time adding two orbits. Because varying number of steps are required for different values of  $\mu$ , we have defined `nsteps` below to accomplish this.

```

t0 = 0.0; h = 0.02; initset = {{0.1, 0}, {-0.1, 0}};
bothdirflag = True;
arrowflag = True; arrowvec = {1/5, 4/5};
parmlist = Module[{ans, i}, ans = {}];
  Do[ans = Append[ans, {i * 0.05}], {i, -40, 20}]; ans];

```

```

nsteps := Module[{ans, mu}, mu = First[parmval];
  If[(mu < -1.45), (ans = 350), (If[(mu < -0.95), (ans = 600),
    (If[(mu < -0.25), (ans = 700),
      (If[(mu < -0.2), (ans = 800),
        (If[(mu < -0.15), (ans = 900),
          (If[(mu < -0.10), (ans = 1000),
            (If[(mu < -0.05), (ans = 1200),
              (If[(mu == 0), (ans = 5000),
                (If[(mu < 0.1), (ans = 1400),
                  (If[(mu < 0.2), (ans = 1200),
                    (If[(mu < 0.25), (ans = 1000),
                      (If[(mu < 0.3), (ans = 900),
                        (If[(mu < 0.35), (ans = 800),
                          (If[(mu < 0.4), (ans = 700),
                            (If[(mu < 0.45), (ans = 600),
                              (ans =
                                500)]])])])])])])])])])])])])])])])])])
]

```

```

initset := Module[{ans, mu}, mu = First[parmval];
  If[((mu < -0.25) || (mu > 0.25)), (ans = {{0.02, 0}, {-0.02, 0}}),
    (ans = {{0.1, 0}, {-0.1, 0}})]; ans]

```

```

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph];

```

```

Bifurcation sequence for parmlist =
{{-2.}, {-1.95}, {-1.9}, {-1.85}, {-1.8}, {-1.75}, {-1.7}, {-1.65},
{-1.6}, {-1.55}, {-1.5}, {-1.45}, {-1.4}, {-1.35}, {-1.3}, {-1.25},
{-1.2}, {-1.15}, {-1.1}, {-1.05}, {-1.}, {-0.95}, {-0.9}, {-0.85},
{-0.8}, {-0.75}, {-0.7}, {-0.65}, {-0.6}, {-0.55}, {-0.5}, {-0.45},
{-0.4}, {-0.35}, {-0.3}, {-0.25}, {-0.2}, {-0.15}, {-0.1}, {-0.05}, {0},
{0.05}, {0.1}, {0.15}, {0.2}, {0.25}, {0.3}, {0.35}, {0.4}, {0.45}, {0.5},
{0.55}, {0.6}, {0.65}, {0.7}, {0.75}, {0.8}, {0.85}, {0.9}, {0.95}, {1.}}

```

