ME 406
Bifurcations III
Transcritical Bifurcation

```
sysid
```

```
Mathematica 4.1.2, DynPac 10.66, 3/5/2002
```

```
intreset; plotreset;
```

■ 1. Introduction

In this notebook, the third in a series of notebooks on bifurcations, we look at a simple example of a transcritical bifurcation. We construct a movie showing the changes of a selected set of orbits with the bifurcation parameter.

■ 2. Definition of the System

We consider the following system, depending on one parameter *a*:

$\dot{x} = \mu x - x^2 \quad , \quad \dot{y} = -y \; .$

This system has two equilibria for any $\mu \neq 0$, and one equilibrium for $\mu = 0$. The bifurcation is $\mu = 0$, for which the two equilibria coalesce at the origin. We begin our analysis by defining the system for DynPac.

```
setstate[{x, y}]; setparm[{μ}]; slopevec = {μ x - x^2, - y};
sysname = "Transcritical Bifurcation";
```

```
eq1 = {0, 0}; eq2 = {μ, 0};
```

Let's look at the nature of the equilibria.

```
eigsys[eq1]
```

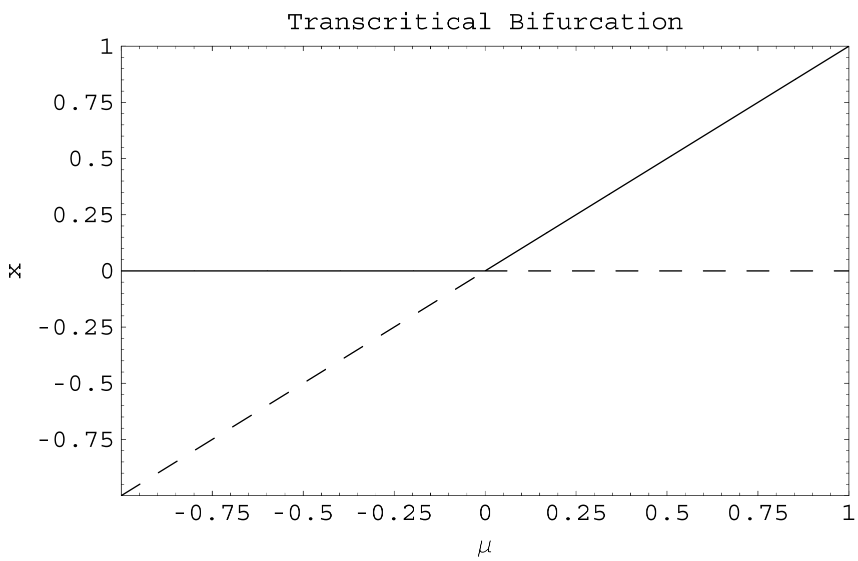
```
{{-1, μ}, {{0, 1}, {1, 0}}}
```

```
eigsys[eq2]
```

```
{{-1, -μ}, {{0, 1}, {1, 0}}}
```

Thus for $\mu < 0$, eq1 is a stable node and eq2 is a saddle. For $\mu > 0$, eq1 is a saddle and eq2 is a stable node. For $\mu = 0$, linearization is inconclusive, although in this case it is easy to integrate the equations directly and show that the equilibrium is unstable. This kind of bifurcation is often called an exchange of stabilities. We may visualize this with a bifurcation diagram, showing the equilibria as functions of μ, with stable in solid, unstable in dashed.

```
plot1 = Plot[{0, μ}, {μ, -1, 0},
   PlotRange → {{-1, 1}, {-1, 1}},
   PlotLabel → "Transcritical Bifurcation",
   FrameLabel → {"μ", "x"}, Axes → False,
   ImageSize → imsize, Frame → True, DisplayFunction → Identity,
   PlotStyle → {Dashing[{0.1, 0}], Dashing[{0.03, 0.03}]}];

plot2 = Plot[{μ, 0}, {μ, 0, 1},
   PlotRange → {{-1, 1}, {-1, 1}},
   PlotLabel → "Transcritical Bifurcation", Frame → True,
   FrameLabel → {"μ", "x"}, Axes → False,
   DisplayFunction → Identity, ImageSize → imsize,
   PlotStyle → {Dashing[{0.1, 0}], Dashing[{0.03, 0.03}]}];

Show[{plot1, plot2}, DisplayFunction → $DisplayFunction];
```



Now we will construct phase plots for various values of the parmeter μ. We first construct a short sequence which will show the essential features, and then we construct a long sequence suitable for a movie. We choose a plotting window of {{-2,2},{-2,2}}, and a set of initial conditions attached to points on the window, plus several initial conditions near the equilibria. The function $f(\mu)$ defined below gives the position of the unstable equilibrium as a function of μ., and $g(\mu)$ gives the position of the stable node.

```
f[μ_] := If[μ > 0, 0, μ]; g[μ_] := If[μ > 0, μ, 0];

ε = 0.02;

initset = {{2, 0}, {2, -1.5}, {2, 1.5}, {0, 0}, {-2, 0},
   {1.5, 2}, {-0.5, 2}, {1.5, -2}, {-0.5, -2}, {-0.5, 2},
   {f[μ], ε}, {f[μ], -ε}, {f[μ] + ε, 0}, {f[μ] - ε, 0}};

plrange = {{-2, 2}, {-2, 2}}; asprat = 1; labshift = 18; imsize = 360;

arrowflag = True; arrowvec = {1 / 2};

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;
```

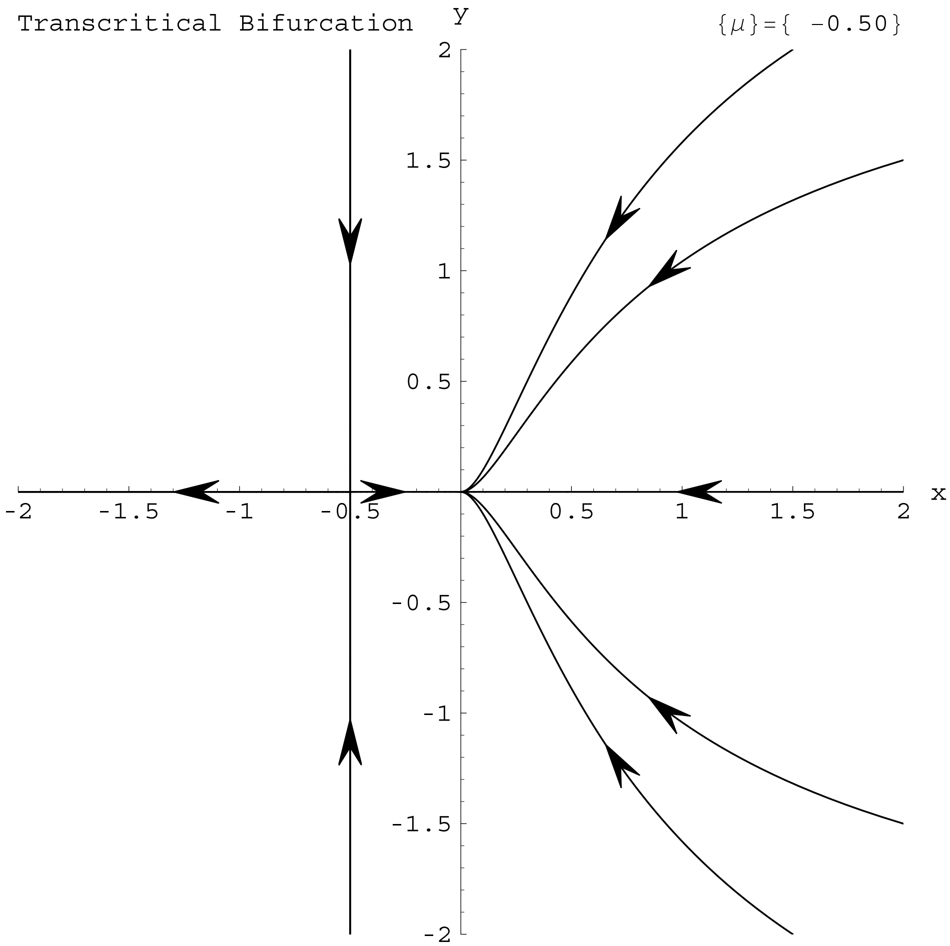
```
rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};
```

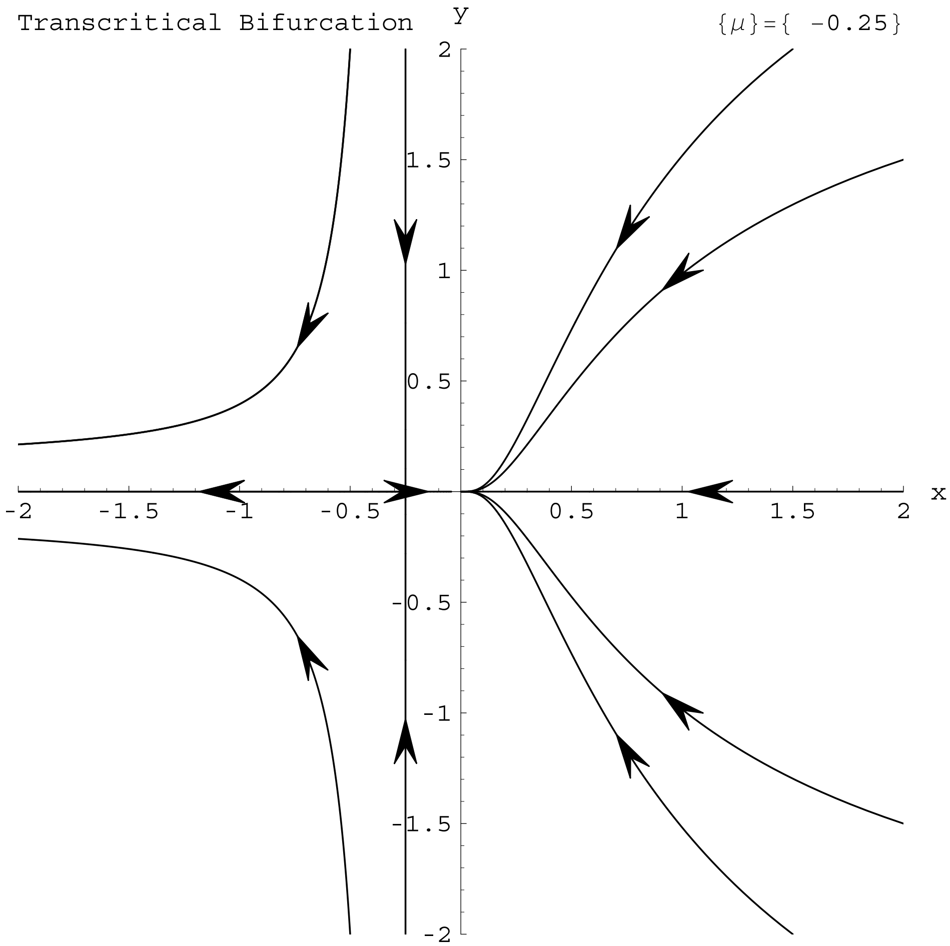
Now we choose a small number of μ-values for a bifurcation sequence.

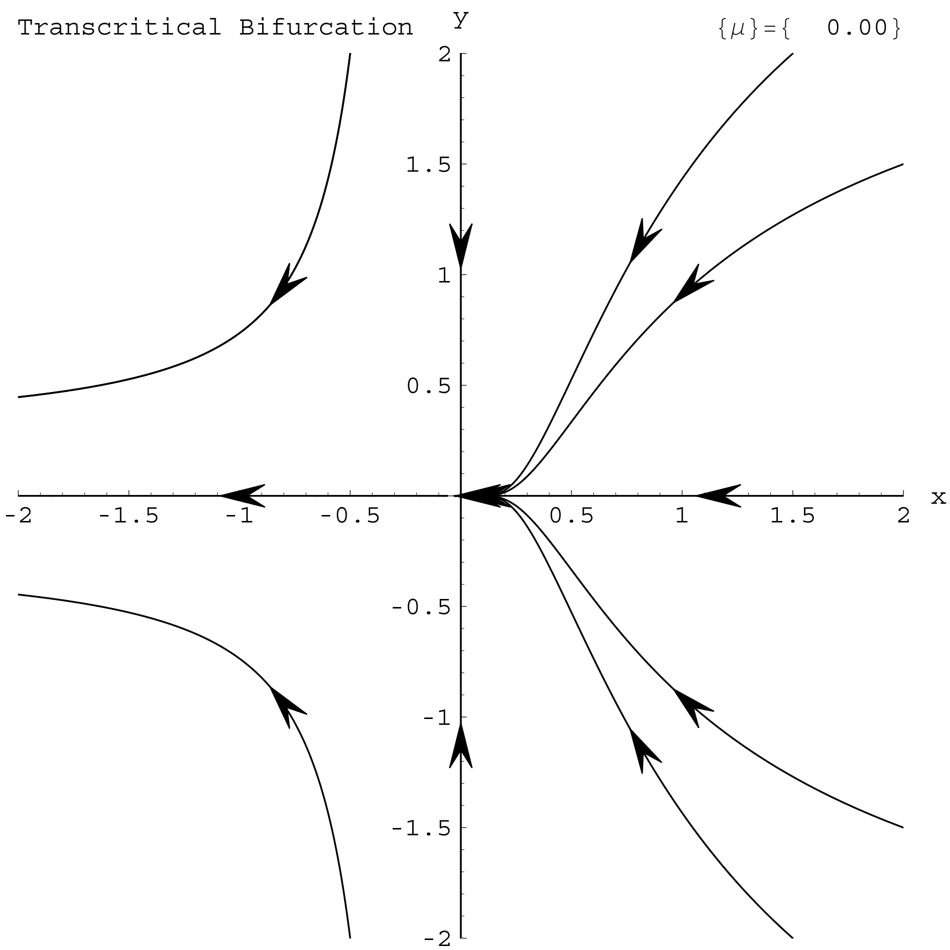
```
parmlist = {{-0.5}, {-0.25}, {0.0}, {0.25}, {0.5}};
```

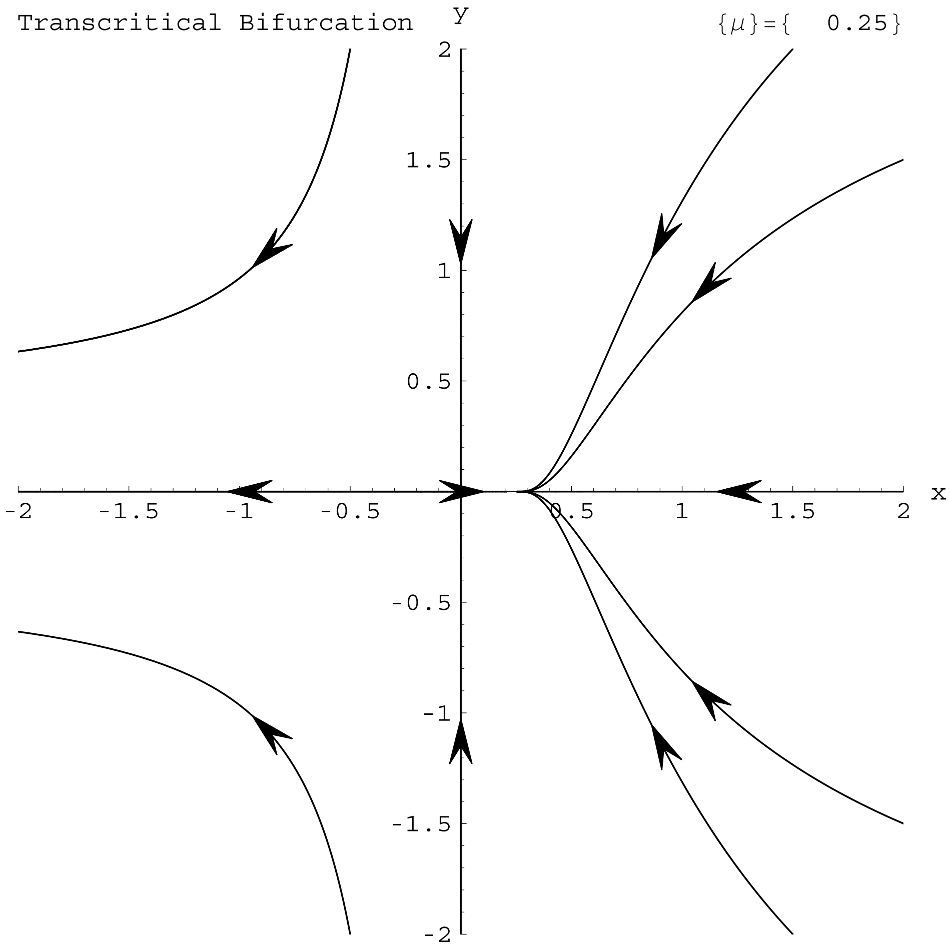
```
bifurc[initset, t0, h, nsteps, 1, 2, parmlist];
```

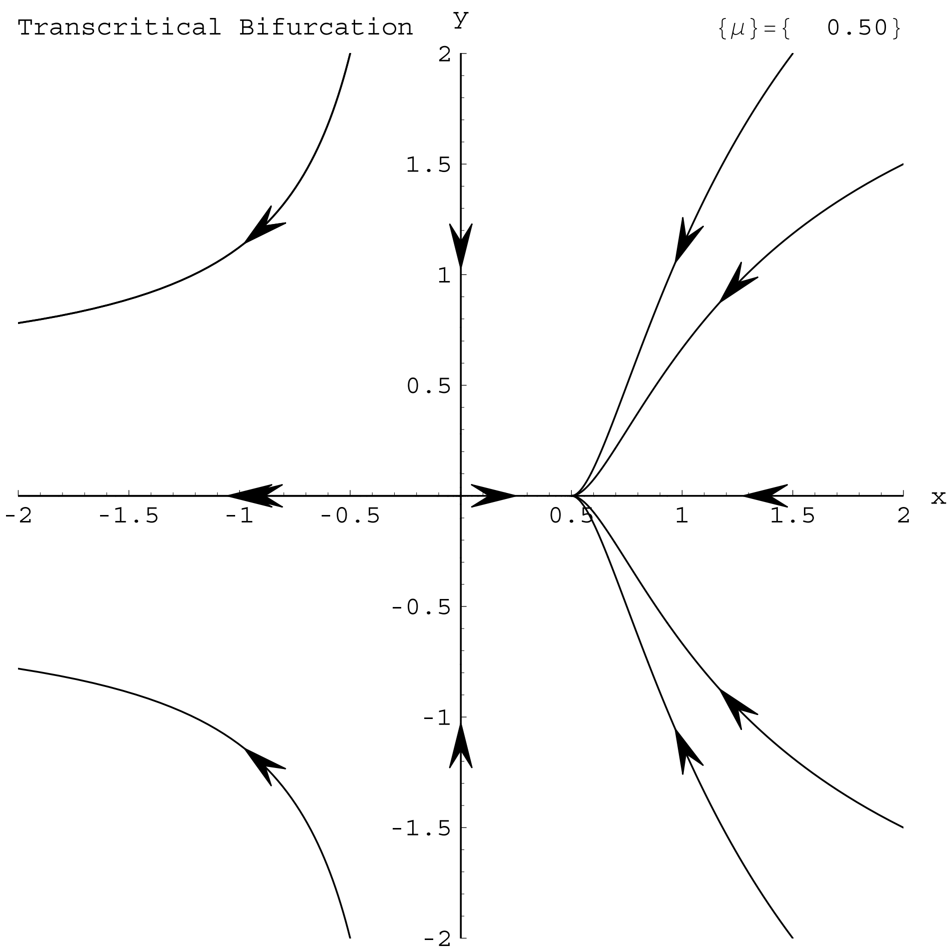
```
Bifurcation sequence for parmlist = {{-0.5}, {-0.25}, {0.}, {0.25}, {0.5}}
```




Transcritical Bifurcation
{μ}={ -0.25}
y
x
2
1.5
1
0.5
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0.5
1
1.5
2


Transcritical Bifurcation
{μ}={ 0.00}
y
x
2
1.5
1
0.5
-0.5
-1
-1.5
-2
-2 -1.5 -1 -0.5 0.5 1 1.5 2


Transcritical Bifurcation
{μ}={ 0.25}
y
x
2
1.5
1
0.5
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0.5
1
1.5
2



The equilibrium for $\mu = 0$ is an interesting composite. It looks like a stable node on the right, and like a saddle on the left.

As our final task in this notebook, we will make a movie of the saddle-node bifurcation. We will concentrate the frames of the movie around $\mu = 0$. We make 41 frames at μ-intervals of 0.015, with μ running from -0.3 to -0.3. We remove the axes for a clearer view, and we place a red dot on the saddle, a blue dot on the stable node. The function refgraph below defines the dot graphs.

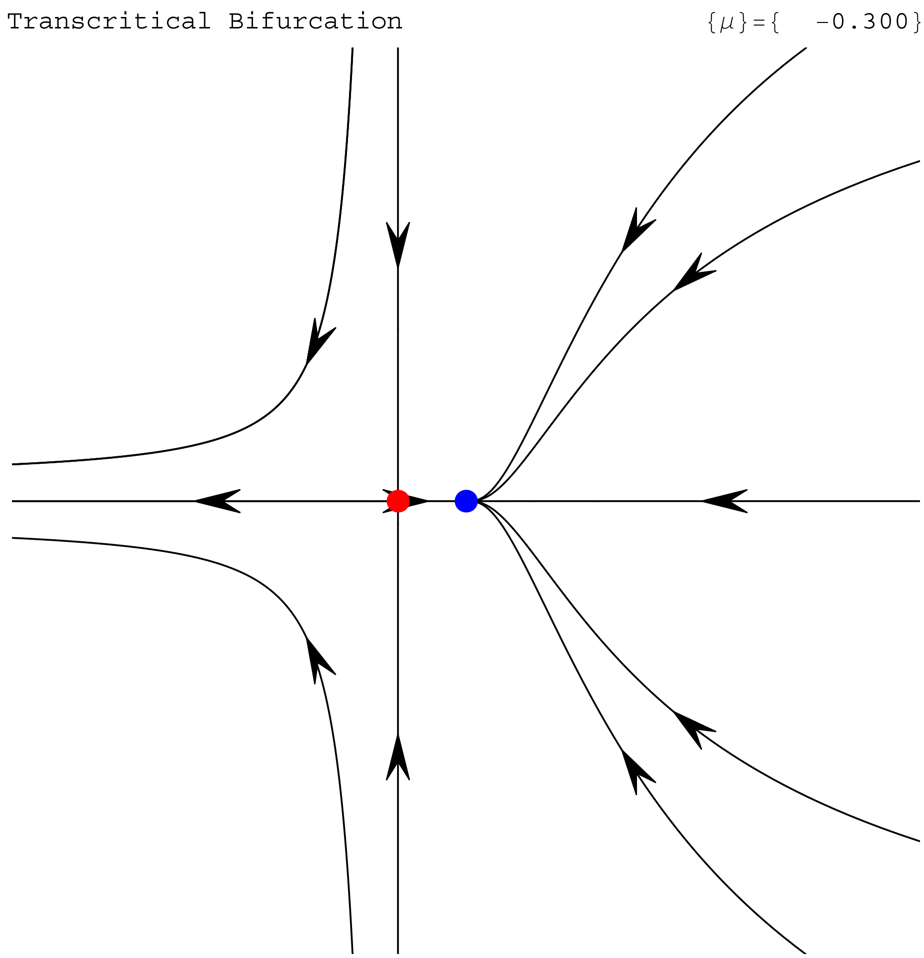
```
refgraph := Module[{temp1, temp2}, display = False; ptsize = 0.025;
  setcolor[{Red}]; temp1 = dots[{{f[μ], 0}}];
  setcolor[{Blue}]; temp2 = dots[{{g[μ], 0}}];
  display = True; setcolor[{Black}]; {temp2, temp1}]
```

```
parmlist = Module[{ans, i}, ans = {};
  Do[ans = Append[ans, {0.015 * i}], {i, -20, 20}]; ans]
```

```
{{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21},
 {-0.195}, {-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12},
 {-0.105}, {-0.09}, {-0.075}, {-0.06}, {-0.045}, {-0.03},
 {-0.015}, {0}, {0.015}, {0.03}, {0.045}, {0.06}, {0.075}, {0.09},
 {0.105}, {0.12}, {0.135}, {0.15}, {0.165}, {0.18}, {0.195},
 {0.21}, {0.225}, {0.24}, {0.255}, {0.27}, {0.285}, {0.3}}
```

```
initset = {{2, 0}, {2, -1.5}, {2, 1.5}, {0, 0}, {-2, 0},
   {1.5, 2}, {-0.5, 2}, {1.5, -2}, {-0.5, -2}, {-0.5, 2},
   {f[μ], ε}, {f[μ], -ε}, {f[μ] + ε, 0}, {f[μ] - ε, 0}};

plrange = {{-2, 2}, {-2, 2}}; asprat = 1;
labshift = 18; imsize = 360; axon  = False;

arrowflag = True; arrowvec = {1 / 2};

totdig = 6; decdig = 3;

t0 = 0.0; h = 0.02; nsteps = 800; bothdirflag = True;

rangeflag = True; ranger = {{-2.1, 2.1}, {-2.1, 2.1}};

bifurc[initset, t0, h, nsteps, 1, 2, parmlist, refgraph];
```

```
Bifurcation sequence for parmlist =
 {{-0.3}, {-0.285}, {-0.27}, {-0.255}, {-0.24}, {-0.225}, {-0.21}, {-0.195},
  {-0.18}, {-0.165}, {-0.15}, {-0.135}, {-0.12}, {-0.105}, {-0.09}, {-0.075},
  {-0.06}, {-0.045}, {-0.03}, {-0.015}, {0}, {0.015}, {0.03}, {0.045},
  {0.06}, {0.075}, {0.09}, {0.105}, {0.12}, {0.135}, {0.15}, {0.165},
  {0.18}, {0.195}, {0.21}, {0.225}, {0.24}, {0.255}, {0.27}, {0.285}, {0.3}}
```


Transcritical Bifurcation
{μ}={ -0.300}