The exam covers all of the material of the course. You may use any books, notes, or references that you like, but you may not exchange material during the exam. You may use calculators but not laptops. Do all six problems. Each problem is worth 18 points. You will get all of the points you earn up to a maximum of 100. Be sure to explain your work. Wrong calculations with no explanation will receive little partial credit. You may pick up a copy of the solutions after the exam. Good luck!

(1) Consider the function $f(x)$ defined below on $[-1, 1]$.

$$ f(x) = \begin{cases} 
  x + 1, & \text{for } -1 \leq x < -\frac{1}{2}, \\
  x, & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\
  \frac{1}{2}, & \text{for } \frac{1}{2} < x \leq 1.
\end{cases} $$

(a) (3 points) Sketch $f(x)$ on the interval $[-1, 1]$. Does $f$ have any discontinuities on this interval? Is $f$ piecewise smooth on this interval?

(b) (5 points) Sketch three periods of the function represented by the Fourier series of $f$, with the base interval of the series being $[-1, 1]$. How fast will the Fourier coefficients drop off with $n$?

(c) (5 points) Sketch three periods of the function represented by the Fourier sine series of $f$ on $[0, 1]$. How fast do the Fourier coefficients drop off with $n$?

(d) (5 points) Sketch three periods of the function represented by the Fourier cosine series of $f$ on $[0, 1]$. How fast do the Fourier coefficients drop off with $n$?

(2) Consider the regular Sturm-Liouville problem given below.

$$ \frac{d}{dx} \left( k(x) \frac{d\Psi}{dx} \right) + \lambda \sigma(x) \Psi = 0, \quad 0 < x < 1, \quad \Psi(0) = 0, \quad \Psi(1) = 0, $$

where the function $k$ is positive and continuously differentiable, and the function $\sigma$ is positive and continuous.

(a) (4 points) State the orthogonality condition satisfied by the eigenfunctions.

(b) (6 points) Prove that the eigenvalues are all positive.

(c) (8 points) Assume that you know all of the eigenfunctions $\Psi_n$ and eigenvalues $\lambda_n$ of this system. Discuss in detail how you would use them to solve the problem given below.

$$ \sigma(x) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k(x) \frac{\partial T}{\partial x} \right), \quad 0 < x < 1, \quad t > 0, $$

with $T(0,t) = 0$, $T(1,t) = 0$, and $T(x,0) = f(x)$. 

(continued next page)
(3) Solve the boundary value problem below for \( \Phi(x,y) \) in a rectangle. Solution by an expansion in appropriate eigenfunctions is recommended, but you may use separation of variables if you prefer.

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,
\]

with \( \Phi(0,y) = 0, \quad \frac{\partial \Phi}{\partial x}(a,y) = \Phi_0 \sin(3\pi y / b), \quad \Phi(x,0) = 0, \quad \text{and} \quad \Phi(x,b) = 0. \)

(4) Use the Fourier transform in \( x \) to solve the boundary value problem given below. Express your answer as a Fourier inversion integral. You do not need to evaluate the integral.

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{e^{-y}}{1 + x^2}, \quad -\infty < x < \infty, \quad 0 < y < \infty, \quad \phi(x,0) = 0, \quad \phi \to 0 \text{ as } y \to \infty.
\]

Hints: \( \int_{-\infty}^{\infty} \frac{e^{-itx}}{1 + x^2} \, dx = \pi e^{-|t|}. \) A particular solution of the equation for \( \Phi \) has the form constant*\( e^{-y} \).

(5) Solve the boundary value problem below for \( \Psi(r,\phi) \), an axisymmetric solution of the Laplace equation, in the region exterior to a sphere of radius \( a \). Here \( r \) is the usual spherical radial coordinate and \( \phi \) is the usual spherical polar angle. For what value of \( b \), a constant appearing in the boundary condition, does \( \Psi \) drop off most rapidly with radius \( r \)? The quantity \( \Psi_0 \) appearing in the boundary condition is a positive constant.

\[
\nabla^2 \Psi = 0, \quad a < r < \infty, \quad 0 \leq \phi \leq \pi,
\]

with \( \frac{\partial \Psi}{\partial r}(a,\phi) = \frac{\Psi_0}{a}[1 + b \cos^2(\phi)], \) and \( \Psi(r,\phi) \to 0 \) as \( r \to \infty. \)

(6) Consider the boundary value problem given below for the Laplace equation in a semi-infinite circular cylinder. There is no \( \theta \)-dependence, so the solution is \( \Phi = \Phi(r,z) \). You are asked to solve this problem by an expansion of the form \( \Phi(r,z) = \sum_{n=1}^{\infty} C_n(z) J_0(\alpha_n(r/a)), \) where \( J_0 \) is the Bessel function of the first kind of order zero, and \( \alpha_n \) is the \( n \)th positive zero of \( J_0 \). Solve the problem by substituting the expansion into the equation to determine the ordinary differential equations satisfied by \( C_n(z) \). In solving the equations for the \( C_n \)'s, you may assume that the boundary function \( f(r) \) has the known expansion \( f(r) = \sum_{n=1}^{\infty} f_n J_0(\alpha_n(r/a)). \)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad 0 < r < a \text{ and } 0 < z < \infty,
\]

with \( \Phi(a,z) = 0, \quad \Phi(r,0) = f(r), \quad \text{and} \quad \Phi \to 0 \text{ as } z \to \infty. \)

Hint: For any constant \( \beta \), \( J_0(\beta r) \) satisfies the equation \( \frac{d}{dr} \left[ r \frac{d}{dr} (J_0(\beta r)) \right] + \beta^2 r J_0(\beta r) = 0. \)