ME 201/MTH 281/ME 400/CHE 400 EXAM #2 SOLUTIONS NOV 12, 2009

(1) (a) $\psi_1(x)$ is possibly correct. It appears to satisfy the BC's, and it has no interior zeros. ψ_2 cannot be correct. It vanishes twice in the interior, but the nth eigenfunction vanishes exactly $n-1$ times in the interior.

(b) To get the Rayleigh quotient we multiply the equation by ψ and integrate over $[0,1]$. We get

$$\psi(x\psi')' + \lambda x^2\psi^2 = 0$$

$$(x\psi\psi')' - x\psi'^2 + \lambda x^2\psi^2 = 0$$

$$x\psi\psi'\Big|_1^2 - \int_1^2 x\psi'^2\,dx + \lambda\int_1^2 x^2\psi^2\,dx = 0$$

The first term vanishes because of the BC's. Then

$$\lambda = \frac{\int_1^2 x\psi'^2\,dx}{\int_1^2 x^2\psi^2\,dx} \geq 0.$$

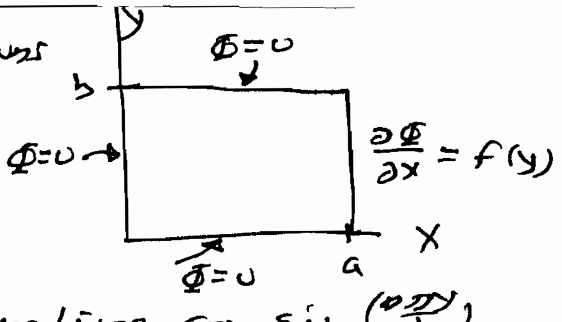
Thus $\lambda_1 = -1$ is impossible.

(c) The Rayleigh quotient shows that $\lambda = 0$ only if $\psi' \equiv 0 \Rightarrow \psi = \text{constant}$, but $\psi(1) = 0 \Rightarrow$ trivial solution. So $\lambda = 0$ is impossible.

(d) False. We know that $\lim_{n\to\infty} \lambda_n = \infty$, whereas the formula gives $\lambda_n \to \frac{1}{3}$ as $n \to \infty$.

(e) The general orthogonality theorem says $\int_1^2 \psi_1 \cdot x^2 \cdot \psi_2\,dx = 0$. This does not tell us about $\int_1^2 \psi_1\psi_2\,dx$. So it is possibly true, possibly false.

(2) (a) From the boundary conditions we see that we need solutions which are oscillatory in y. This rules out (1). We also want Φ to vanish at $y=0$ and $y=b$. The appropriate y-functions are $\sin\left(\frac{n\pi y}{b}\right)$, so (2) is the correct form.



(b) We substitute $\Phi(x,y) = \sum G_n(x)\sin\left(\frac{n\pi y}{b}\right)$ into the equation to get

$$\sum\left(\frac{d^2G_n}{dx^2} - \frac{n^2\pi^2}{b^2}G_n\right)\sin\left(\frac{n\pi y}{b}\right) = 0.$$

(2) (b) (continued). Then $\frac{d^2 G_n}{dx^2} - \frac{n^2\pi^2}{b^2} G_n = 0$, so

$$G_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right).$$

We impose the homogeneous boundary condition:

$$G_n(0) = 0 = A_n, \text{ so } G_n(x) = B_n \sinh\left(\frac{n\pi x}{b}\right).$$

We now have

$$\Phi(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right).$$

We impose the inhomogeneous BC on $x = a$:

$$\left.\frac{\partial \Phi}{\partial x}\right|_{x=a} = f(y) = C\sin\left(\frac{3\pi y}{b}\right) = \sum \frac{n\pi}{b} B_n \cosh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$\text{so } \quad C\sin\left(\frac{3\pi y}{b}\right) = \sum\left(\frac{n\pi}{b} B_n \cosh\left(\frac{n\pi a}{b}\right)\right)\sin\left(\frac{n\pi y}{b}\right).$$

Equating coefficients, we get $B_n = 0$ for $n \neq 3$, and

$$\frac{3\pi}{b} B_3 \cosh\left(\frac{3\pi a}{b}\right) = C$$

$$\text{so } \quad B_3 = \frac{bC}{3\pi \cosh\left(\frac{3\pi a}{b}\right)}$$

and

$$\Phi(x,y) = \frac{bC \sinh\left(\frac{3\pi x}{b}\right)}{3\pi \cosh\left(\frac{3\pi a}{b}\right)} \sin\left(\frac{3\pi y}{b}\right).$$

(3) We look for standing modes: $P = \cos(\omega t)\,\Phi$. Substitute into the equation to get $\frac{d^2\phi}{dx^2} + k^2\phi = 0$, where $k^2 = \omega^2/c^2$. The general solution for ϕ is $\phi = A\cos kx + B\sin kx$. The boundary condition at $x = 0$ is $d\phi/dx = 0 \Rightarrow B = 0$. At $x = L$, $\phi = 0 \Rightarrow \cos kL = 0$. Then $kL = (n - \frac{1}{2})\pi$, $n = 1, 2, 3, \ldots$. For the fundamental $n = 1$, so $kL = \pi/2$. Then $\omega = ck = \frac{\pi c}{2L}$. The frequency in Hz is $\nu = \frac{\omega}{2\pi} = \frac{c}{4L} = \frac{(330)\,(m/s)}{4\,(0.25)\,m} = 330$ Hz.

(4) (a) The right-hand side of the given equation is ≤ 0. Thus $\int_0^L \frac{1}{2}\hat{T}^2 dx$ is non-increasing. Because it is positive definite, it can never be less than zero. At $t=0$, $\hat{T}=0$, so the integral is zero. Thus it starts at zero, cannot be negative, and is non-increasing. Therefore it stays zero, hence $\hat{T} \equiv 0$ for all time.

(b) When $\alpha < 0$, the rhs can be either negative or positive, so we can no longer draw the conclusion that $\int_0^L \frac{1}{2}\hat{T}^2 dx$ does not increase, and our proof breaks down.

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DETAILED SOLUTIONS AND COMMENTS

PROBLEM 1 OF EXAM #2

Solutions

(a) As near as one can tell, the functions in the graphs do satisfy the boundary conditions. According to the oscillation theorem, the nth eigenfunction should have $n-1$ interior zeros. This condition is satisfied by ψ_1 but not by ψ_2. Hence the graph of ψ_1 is possibly correct whereas the graph of ψ_2 is necessarily incorrect.

Some of you worried that ψ_1 was not sufficiently oscillatory. Consider the example $\psi'' + \lambda\psi = 0$, $1<x<2$, $\psi(1)=0$, $\psi'(2)=0$. The first eigenfunction is $\psi_1 = \sin[\frac{1}{2}\pi(x-1)]$ which has a graph very similar to ψ_1.

(b) The Rayleigh Quotient is the key here. Many of you who tried that had an incorrect derivation. We have $(x\psi')' + \lambda x^2\psi = 0$, so $\psi(x\psi')' + \lambda x^2\psi^2 = 0$. We integrate by parts to get

$$(x\psi\psi')' - x\psi'^2 + \lambda x^2\psi^2 = 0$$

Now integrate over $[1,2]$:

$$(x\psi\psi')\Big|_1^2 - \int_1^2 x\psi'^2\,dx + \lambda\int_1^2 x^2\psi^2\,dx = 0.$$

The integrated term vanishes because of the boundary conditions. Then

$$\lambda = \frac{\int_1^2 x\psi'^2\,dx}{\int_1^2 x^2\psi^2\,dx} \geq 0.$$

Thus $\lambda = -1$ is impossible.

(c) If $\lambda = 0$, the Rayleigh quotient tells us that $\psi' = 0$ everywhere. Then $\psi = \text{constant}$. But $\psi(1) = 0$, so constant $= 0$ and solution is trivial. $\therefore \lambda \neq 0$.

(c) (continued) An alternative argument is that $\lambda \geq 0$ and $\lambda_1 < \lambda_2$ implies $\lambda_2 > 0$.

(d) This formula is wrong for several reasons: it predicts $\lambda_1 = -1 < 0$, it predicts $\lambda_2 = 0$, and it is not consistent with the general requirement that $\lambda_n \to \infty$ as $n \to \infty$.

Several of you said the formula is wrong because it does not contain π. That is not a valid argument. Consider the example $\psi'' + \lambda\psi = 0$ on $0 < x < \pi$, $\psi(0) = 0$, $\psi(\pi) = 0$. It is easy to show that the eigenvalues are $\lambda_n = n^2$, $n = 1, 2, 3 \ldots$

(e) Orthogonality tells us that $\int_1^2 \psi_m(x) P(x) \psi_n(x) = 0$ when in this problem $P(x) = x^2$. Thus we know that

$$\int_1^2 \psi_1 \cdot x^2 \cdot \psi_2 \, dx = 0.$$

We get no conclusion about $\int_1^2 \psi_1(x) \psi_2(x) dx$, which may or may not be zero.

A number of you claimed that the equation had a solution in the form of sines and cosines. That is not correct for an equation with variable coefficients. Some of you also claimed that the eigenvalues had the form $n^2\pi^2/L^2$ - again not correct for this problem. As we showed above, it is perfectly possible to answer the questions posed without solving the equation. To do so requires a basic knowledge of Sturm-Liouville systems.

(The general solution of the equation is

$$\psi(x) = C_1 J_0\left(\tfrac{2}{3}x^{3/2}\right) + C_2 Y_0\left(\tfrac{2}{3}x^{3/2}\right),$$ where J_0 and Y_0 are Bessel functions.)