ME 201/MTH 281/ME400/CHE400
EXAM #2 REVIEW 2009

The exam will be on Thursday November 12 from 2:00 to 3:15 in Dewey 1101 and from 3:25 to 4:40 in Meliora 221, the regular Thursday classrooms. The material in the class notes which will be covered on the exam is sections 3.2 – 3.6 from chapter 3, all of chapters 4 and 5, and sections 6.1 and 6.2 from chapter 6. The exam will cover homework assignments 5, 6, 7 and 8. The exam will be open book and open notes -- any reference material is allowed. The class on Wednesday November 11 will be a review class.

PRACTICE EXAM

For a practice exam, you can use exam #2 given in 2008, which is available, with solutions, on the web. After you have completed your basic review, you should try to work that exam in 75 minutes.

SUGGESTED REVIEW PROBLEMS

The purpose of the problems below is to provide a review of the main concepts and techniques to be covered on this exam. Some of the problems are short, and some are longer than would be reasonable on an exam. You should work these problems and also review all of the homework problems. I will answer any questions you have in class on Wednesday November 11.

SEPARATION OF VARIABLES, PART 1

Laplace Equation

(1) Find the solution of the boundary-value problem given below.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0, \quad 0<x<a, 0<y<b,$$

with $\Phi(x,0)=0$, $\Phi(x,b)=0$, $\Phi(0,y)=0$,

and $\frac{\partial\Phi}{\partial x}(a,y)=\Phi_1\sin\left(\frac{\pi y}{b}\right)+\Phi_3\sin\left(\frac{3\pi y}{b}\right)$,

where Φ_1 and Φ_3 are constants.

Answer: $\Phi(x,y)=\Phi_1\frac{\sinh(\pi x/b)}{(\pi/b)\cosh(\pi a/b)}\sin\left(\frac{\pi y}{b}\right)+\Phi_3\frac{\sinh(3\pi x/b)}{(3\pi/b)\cosh(3\pi a/b)}\sin\left(\frac{3\pi y}{b}\right)$.

Wave Equation

(2) If a stretched string is elastically bound to its equilibrium position at each point, the governing partial differential equation takes the form

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}-ky \quad ,$$

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where k is a positive constant. (This equation is called the Klein-Gordon equation.) Suppose the string is fixed at the ends $x = 0$ and $x = L$. Find the frequencies of vibration and show that the frequencies are no longer simple integral multiples of the frequency of the fundamental mode.

Answer: $\omega_n = \left[\left(\frac{n\pi c}{L}\right)^2 + k\right]^{1/2}$.

Energy Integrals and Uniqueness

(3) Prove that the solution is unique for the boundary-value problem given below for a Laplace-like equation in a volume V bounded by a closed surface S. The function Γ is a known function of position in V, and the function f is a known function of position on S.

$$\nabla^2\Phi - \alpha\Phi = \Gamma \text{ in V} \ , \ \Phi|_S = f \ .$$

Here α is a positive constant. Is your proof valid when α is negative?

STURM-LIOUVILLE THEORY

(4) Consider the Sturm-Liouville system given below. Prove that there are no negative eigenvalues. Find the eigenvalues and eigenfunctions. Expand the function $f(x) = x$ in a series of the eigenfunctions.

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad , \quad 0 < x < 1 \quad , \quad \text{with} \quad \frac{dy}{dx}(0) = 0 \quad \text{and} \quad \frac{dy}{dx}(1) = 0 \quad .$$

Answer: $x = \frac{1}{2} - \sum_{n\ odd} \frac{4}{(n\pi)^2}\cos(n\pi x)$.

(5) Let $\{\phi_n(x)\}$ be the set of normalized eigenfunctions of a regular Sturm-Liouville system on [a,b]. Let $p(x)$ be the weight function of the system. Show that for any functions f and g, continuous and piecewise smooth on [a,b], it is true that

$$\int_a^b pfg\,dx = \sum_{n=1}^{\infty}(pf,\phi_n)(pg,\phi_n) \ .$$

(Interpretation: The "dot product" of the "vectors" f and g is equal to the sum of the products of corresponding components.)

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SEPARATION OF VARIABLES, PART 2

(6) Find the solution of the boundary value problem given below.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0, \quad 0<x<a, 0<y<b,$$

$$\text{with } \Phi(x,0)=0, \frac{\partial\Phi}{\partial y}(x,b)=1, \frac{\partial\Phi}{\partial x}(0,y)=0, \Phi(a,y)=0 .$$

Answer: $\Phi(x,y)=\sum_{n=1}^{\infty}\frac{2a(-1)^{n+1}}{\pi^2(n-\frac{1}{2})^2}\frac{\sinh[(n-\frac{1}{2})\pi y/a]}{\cosh[(n-\frac{1}{2})\pi b/a]}\cos[(n-\frac{1}{2})\pi x/a]$.

(7) A stretched string acted on by a distributed force $F(x,t)$ per unit length satisifies the equation

$$\frac{\partial^2 y}{\partial t^2}=C^2\frac{\partial^2 y}{\partial x^2}+\frac{F}{\sigma},$$

where σ is the mass per unit length of the string and $C^2=T/\sigma$, with T being the tension in the string. We take the usual boundary conditions of zero displacement at the ends $x=0, L$ of the string. Use an appropriate eigenfunction expansion to solve for $y(x,t)$ when the initial displacement and velocity are zero, and when

$$F(x,t)=F_0 e^{-\alpha t}\sin(3\pi x/L),$$

where F_0 and α are positive constants.

Answer: $y(x,t)=\frac{F_0\sin(3\pi x/L)}{\sigma(\alpha^2+\omega^2)}\left[e^{-\alpha t}-\cos(\omega t)+\frac{\alpha}{\omega}\sin(\omega t)\right]$, where $\omega=\frac{3\pi C}{L}$.

(8) Consider a nuclear reactor with a slab geometry. The neutron density $N(x,t)$ satisfies

$$\frac{\partial N}{\partial t}=D\frac{\partial^2 N}{\partial x^2}+\alpha N-\beta N, \quad 0<x<L .$$

Suppose the boundary conditions are $N(0,t)=0$ and $\frac{\partial N}{\partial x}(L,t)=0$. Find the critical size of the reactor. Answer: $L=\frac{\pi}{2}\sqrt{\frac{D}{\alpha-\beta}}$.

(9) Solve the initial value problem given below for transient heat conduction in two space dimensions.

$$\frac{\partial T}{\partial t}=D\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right), \quad 0<x<a, \quad 0<y<b, \quad t>0,$$

with $T(x,0,t)=0, \quad T(a,y,t)=0, \quad T(x,b,t)=0, \quad T(0,y,t)=0$,

and $T(x,y,0)=T_0$, a constant.

Answer: $T(x,y,t)=\frac{16T_0}{\pi^2}\sum_{m,n=1\text{ (odd)}}^{\infty}\frac{1}{mn}\exp\left(-D\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right]t\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$.

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FOURIER TRANSFORMS

(10) (a) Find the Fourier transform of the function

$$f(x) = \begin{Bmatrix} 0 \text{ for } x < 0\ , \\ e^{-ax} \text{ for } x \geq 0 \end{Bmatrix},$$

where a is a positive constant.

(b) Find the Fourier transform of the function

$$g(x) = \begin{Bmatrix} e^{ax} \text{ for } x < 0\ , \\ 0 \text{ for } x \geq 0 \end{Bmatrix},$$

where a is a positive constant.

(c) Use your results of parts (a) and (b) to find the Fourier transform of $h(x) = e^{-a|x|}$ where a is a positive constant.

Answers: (a) $1/(a + ik)$, (b) $1/(a - ik)$, (c) $2a/(a^2 + k^2)$