ME 201/MTH 281/ME 400/CHE 400 EXAM #2

THURSDAY NOVEMBER 12, 2009 2:00 – 3:15 PM and 3:25 – 4:40 PM

This exam covers homework assignments 5 through 8, and the following sections of the class notes: sections 3.2 through 3.6 of Chapter 3, Chapters 4 and 5, and sections 6.1 and 6.2 of Chapter 6. You may use any books, notes or reference material that you like, but you may not exchange reference material with anyone else. If you need additional information to work a problem, ask me for it, and, if it is appropriate, I will put it on the board. Do all four problems. The value of each is shown, and the total possible is 100. **BE SURE TO EXPLAIN YOUR WORK!** Wrong calculations with no explanation will receive very little partial credit. Solutions will be posted on the web when both exams are over. Graded exams will be returned in class on Monday. Good luck!

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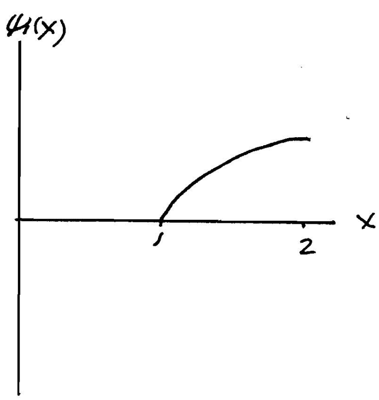
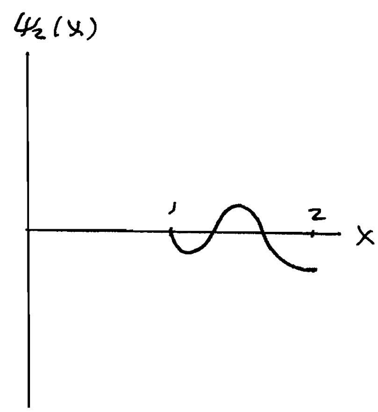
(1) (25 points) You are testing μ-Soft's latest product – an analyzer for Sturm-Liouville systems called EigenVista. Before buying this software, you test it out on the following regular Sturm-Liouville:

$$\frac{d}{dx}\left[x\frac{d\Psi}{dx}\right]+\lambda x^2\Psi=0\ ,1<x<2,\ \text{with } \Psi(1)=0 \text{ and } \frac{d\Psi}{dx}(2)=0\ .$$

Some results from EigenVista are given below. In each case tell whether the results are necessarily true, possibly true or possibly false, or necessarily false. Give your reasoning. Each part is worth 5 points.

(a) When asked for graphs of the first two eigenfunctions, EigenVista returns





(b) When asked for the first eigenvalue, EigenVista returns $\lambda_1=-1$.

(c) When asked for the second eigenvalue, EigenVista returns $\lambda_2=0$.

(d) The general formula for the eigenvalues is $\lambda_n=\frac{1}{3}\left(1-\frac{4}{n^2}\right)$.

(e) $\int_1^2\psi_1(x)\psi_2(x)\,dx=0$, where ψ_1 and ψ_2 are the first two eigenfunctions.

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(2) (30 points) Consider the boundary value problem given below for the Laplace equation in a rectangle.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \quad , \quad 0<x<a \text{ and } 0<y<b \text{ , with } a\neq b,$$

$$\text{and with } \Phi(0,y)=0 \text{ , } \Phi(x,0)=0 \text{ , } \Phi(x,b)=0 \text{ , and } \frac{\partial\Phi}{\partial x}(a,y)=f(y).$$

(a) (10 points) Which of the expansions given below has the appropriate form for the solution of this problem? Explain your answer.

$$(1) \quad \Phi(x,y)=\sum_{n=1}^{\infty}F_n(y)\sin\left(\frac{n\pi x}{a}\right),$$

$$(2) \quad \Phi(x,y)=\sum_{n=1}^{\infty}G_n(x)\sin\left(\frac{n\pi y}{b}\right),$$

$$(3) \quad \Phi(x,y)=\sum_{n=1}^{\infty}H_n(x)\sin\left(\frac{n\pi y}{a}\right).$$

(b) (20 points) By using the correct expansion from part (a), solve the problem when $f(y)=C\sin\left(\frac{3\pi y}{b}\right)$ where C is a positive constant.

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(3) (25 points) An organ pipe of length L is open at the end $x = L$ and closed at the end $x = 0$. The pressure variations P in the pipe satisfy the equation and boundary conditions given below. The quantity C in the equation is the speed of sound in the air in the pipe. Find the fundamental frequency in Hz for the pipe when $L =$ 0.25 m and C = 330 m/s. Do this by looking for a standing wave solution of the form $P=\cos(\omega t)\phi(x)$.

$$\frac{\partial^2 P}{\partial t^2}=C^2\frac{\partial^2 P}{\partial x^2}, \quad 0<x<L \text{ , } t>0 \text{ , } \quad \text{with } \frac{\partial P}{\partial x}(0,t)=0 \text{ , } P(L,t)=0 \text{ .}$$

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(4) (20 points) In this problem, you will establish uniqueness for a solution of the heat equation. In order to keep the problem at a reasonable length, you will be given the setup for the uniqueness proof, and asked only to supply the logic of the final steps. Here is the original problem. The quantities D, k, and h are all positive constants, T_L and T_A are given constants, and $f(x)$ is a given function.

$$\frac{\partial T}{\partial t}=D\frac{\partial^2 T}{\partial x^2} \text{ , } 0<x<L \text{ , } t>0 \text{ ,}$$

$$\text{with } T(0,t)=T_L \text{ , } k\frac{\partial T}{\partial x}(L,t)+hT(L,t)=hT_A \text{ ,}$$

$$\text{and } T(x,0)=f(x).$$

To prove uniqueness, we consider the difference of two solutions, $\hat{T}(x,t)=T_1(x,t)-T_2(x,t)$.

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(4) (continued) We must show that $\hat{T}$ is identically zero. The problem satisfied by $\hat{T}$ is

$$\frac{\partial \hat{T}}{\partial t} = D\frac{\partial^2 \hat{T}}{\partial x^2}\ ,0 < x < L\ ,t > 0\ ,$$

with $\hat{T}(0,t) = 0$, $k\dfrac{\partial \hat{T}}{\partial x}(L,t) + h\hat{T}(L,t) = 0$,

and $\hat{T}(x,0) = 0$.

By multiplying the equation by $\hat{T}$ and integrating with respect to *x* from 0 to *L*, one can derive the equation given below. You may take this as given.

$$\frac{d}{dt}\int_0^L \frac{1}{2}\hat{T}^2\,dx = -D\int_0^L \left(\frac{\partial \hat{T}}{\partial x}\right)^2 dx - \alpha D\left[\hat{T}(L,t)\right]^2,\text{ where } \alpha = \frac{h}{k}.$$

(a) (15 points) Use this equation to prove uniqueness.

(b) (5 points) Does your proof work when $\alpha < 0$? Explain your answer.