ME 201/MTH 281/ME 400/CHE 400 EXAM #1 SOLUTIONS OCT 9 2009

(1) We try $T = F(x)G(t)$. We substitute this into the equation and divide by FG to get

$$\frac{1}{G}\frac{dG}{dt} = \frac{D(x)}{F}\frac{d^2F}{dx^2} - \delta e^{-\beta t}.$$

We move the last term on the right to the other side to get

$$\frac{1}{G}\frac{dG}{dt} + \delta e^{-\beta t} = \frac{D(x)}{F}\frac{d^2F}{dx^2}.$$

The variables are now separated, so each side is equal to the same constant, which we call $-\lambda$. Then

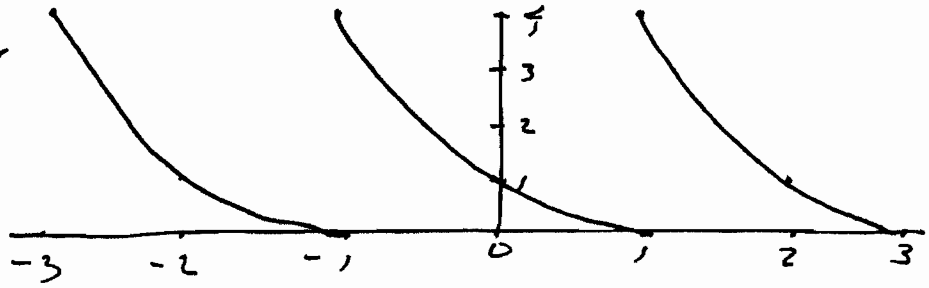
$$D(x)\frac{d^2F}{dx^2} + \lambda F = 0, \quad 0 < x < L$$

and

$$\frac{dG}{dt} + (\lambda + \delta e^{-\beta t})G = 0, \quad t > 0.$$

We impose on F the same homogeneous boundary conditions as those satisfied by T: $F(0)=0$, $F'(L)=0$.

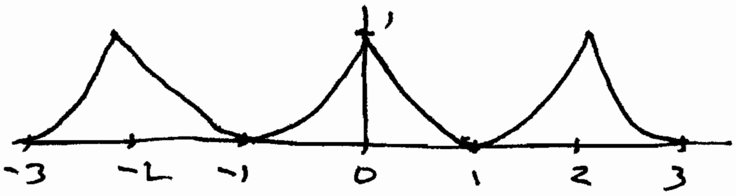
(2) (a) The extended function is discontinuous, so the convergence of the Fourier series will be like $1/n$.



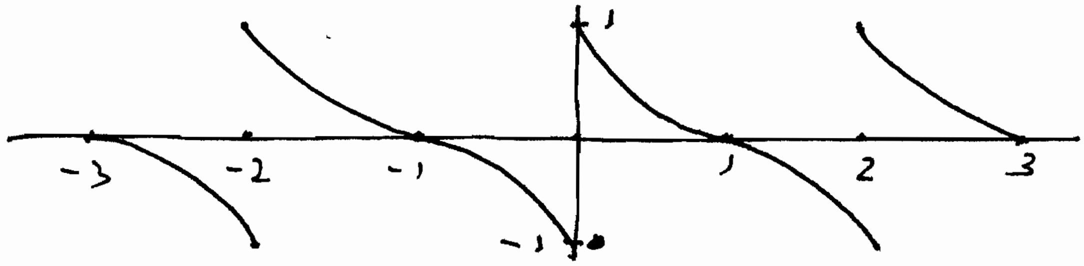
(b) For the cosine series, we want the periodic extension of the even extension. The even extension is

$$f_e(x) = \begin{cases} (1-x)^2 & \text{on } 0 \le x \le 1 \\ (1+x)^2 & \text{on } -1 \le x \le 0 \end{cases}$$

The extended function is continuous, but has discontinuities in slope at $0, \pm 2, \pm 4, ..$. Hence convergence like $1/n^2$.



(2) (continued) (c) For the sine series, we want the periodic extension of the odd extension. The odd extension is $f_o(x) = \begin{cases} f(x), & 0<x<1 \\ -f(-x), & -1<x<0 \end{cases} = \begin{cases} (1-x)^2, & 0<x<1 \\ -(1+x)^2, & -1<x<0. \end{cases}$



The extended function is discontinuous, so the convergence will be like $1/n$.

(3) We must split the solution into a transient ($\hat{T}$) and steady-state (T_s) parts. We start with T_s. It satisfies $D\dfrac{d^2 T_s}{dx^2} + \gamma = 0$. We integrate twice to get $T_s = -\dfrac{\gamma x^2}{2D} + Ax + B$. The boundary condition $T_s(0) = 0$ gives $B = 0$. The boundary condition $T_s(L) = 0$ gives $A = \dfrac{\gamma L}{2D}$, so $T_s(x) = \dfrac{\gamma}{2D} x(L-x)$.

The transient $\hat{T}$ satisfies the homogeneous equation

$$\frac{\partial \hat{T}}{\partial t} = D \frac{\partial^2 \hat{T}}{\partial x^2}, \quad 0<x<L, \; t>0,$$

the boundary conditions $\hat{T}(0,t) = 0$, $\hat{T}(L,t) = 0$, and the initial condition

$$\hat{T}(x,0) = T(x,0) - T_s(x) = T_0 \sin\left(\frac{\pi x}{L}\right).$$

In class we found the solution for $\hat{T}(x,t)$ by separation of variables. The form of the solution was $\hat{T}(x,t) = \sum_{n=1} C_n e^{-n^2\pi^2 Dt/L^2} \sin(n\pi x/L)$.

(3) (continued) The initial condition requires

$$\hat{T}(x,0) = T_0 \sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin(n\pi x/L).$$

By balancing coefficients we see $C_1 = T_0$, and $C_n = 0$ for $n>1$. The solution is then

$$T(x,t) = \frac{\gamma}{2D} x(L-x) + T_0 e^{-\pi^2 Dt/L^2} \sin\left(\frac{\pi x}{L}\right).$$

(4) As we have learned from examples in class, we may estimate the time required for significant diffusion by the diffusion time, which is

$$\tau \simeq a \frac{L^2}{D}.$$

Here L is the length over which the diffusion takes place, D is the diffusivity, and a is a dimensionless constant depending on the geometry. For a slab, $a = \frac{1}{\pi^2}$ and we use that value here. radius of ball

$$\tau = \frac{L^2}{\pi^2 D} = \frac{(2.5\times10^{-2})^2}{\pi^2(5\times10^{-8})} = 1.267\times10^3 s$$
$$= 21 \text{ minutes}$$

If we omit the π^2, this becomes 10 times larger. With a heating time of 21 minutes, a two-minute soaking will not do much. The process is not effective. By techniques we will learn later, we can estimate that in two minutes, the heating will penetrate only about $\frac{1}{4}$ cm.