ME201/MTH281/ME400/CHE400
EXAM #1 REVIEW 2009

The exam will be on Thursday October 8 from 2:00 to 3:15 in Dewey 1101 and 3:25 – 4:40 in Meliora 221, the regular Thursday classrooms. It will cover chapters 1 and 2 and section 3.1 of the class notes, and homework assignments 1, 2, 3, and 4, although there will be no direct questions on the review material of Assignment #1. The exam will be open book and open notes with any reference material allowed.

PRACTICE EXAM

For a practice exam, you can use the exam given in 2008, which is available, with solutions, on the web. After you have completed your basic review, you should try to work that exam in 75 minutes.

SUGGESTED REVIEW PROBLEMS

The purpose of the problems below is to provide a review of the main concepts and techniques to be covered on this exam. Some of the problems are short, and some are longer than would be reasonable on an exam. You should work these problems and also review all of the homework problems. We will use the class on Wednesday Oct. 7 to go over any questions you have on this material.

BALANCE EQUATIONS

(1) Write in integral form a mass balance equation for a substance which diffuses in the atmosphere, and which is also consumed at a known rate by a chemical reaction. Define all the terms you introduce and specify their dimensions. Convert your integral balance to a partial differential equation.

SEPARATION OF VARIABLES

(2) Consider the initial value problem for $\Psi(x,t)$ specified by

$$\frac{\partial \Psi}{\partial t} + U\frac{\partial \Psi}{\partial x} = D\frac{\partial^2 \Psi}{\partial x^2},\ 0 < x < L,\ t > 0,$$

$$\text{with } \Psi(0,t) = 0,\ \frac{\partial \Psi}{\partial x}(L,t) = 0,\ \text{and } \Psi(x,0) = f(x).$$

Here U and D are positive constants. Look for separated solutions in the form $\Psi = F(x)G(t)$ and find the equations satisfied by F and G, as well as the boundary conditions satisfied by F. You do not need to solve the equation for F. (Answer: $F'' - \frac{U}{D}F' + \lambda F = 0$. The equation for Ψ describes one-dimensional diffusion in the x-direction in a fluid which is moving with a constant velocity U, also in the x-direction.

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BASIC FOURIER SERIES

(3) Find the Fourier series on $-1 \le x \le 1$ of the function $f(x) = \sin(x)$. Sketch the periodically extended function represented by the series. (Answer: $2\sin(1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\pi}{n^2\pi^2-1}\sin(n\pi x)$.)

CONVERGENCE OF FOURIER SERIES

(4) Find an example of a polynomial of degree 4 on $-1 \le x \le 1$ which has a Fourier series that may be differentiated termwise three times but not four times.

ORTHOGONALITY

(5) Let **i** and **j** be the usual orthogonal unit vectors in a rectangular *x-y* coordinate system. Let $\mathbf{E}_1 = \mathbf{i} + \mathbf{j}$. Find a vector $\mathbf{E}_2$ orthogonal to $\mathbf{E}_1$. Let **R** be an arbitrary vector in the plane. Use orthogonality to find the components of **R** with respect to $\mathbf{E}_1$ and $\mathbf{E}_2$. In particular, carry this out for the vector $\mathbf{R} = 3\mathbf{i} - 4\mathbf{j}$. (One answer: $\mathbf{R} = -\frac{1}{2}\mathbf{E}_1 + \frac{7}{2}\mathbf{E}_2$, with $\mathbf{E}_2 = \mathbf{i} - \mathbf{j}$.)

FOURIER SINE AND COSINE SERIES

(6) For each of the functions given below on $0 \le x \le 1$, sketch the periodically extended function represented by (1) the Fourier sine series, and (2) the Fourier cosine series. In each case, discuss the convergence properties of the series (i.e., how fast the coefficients drop with *n*).

(a) $f(x) = 2 + 3x$ (Answer: sine like $1/n$, cosine like $1/n^2$.)
(b) $f(x) = x(1-x)$ (Answer: sine like $1/n^3$, cosine like $1/n^2$.)
(c) $f(x) = \sin(\pi x)$ (Answer: convergence not an issue with sine series, cosine like $1/n^2$.)

SEPARATION OF VARIABLES REVISITED

(7) Find a simple long-time approximation for the solution of the initial value problem given below.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}\ , 0 < x < L\ , t > 0\ ,$$

The boundary conditions are $T(0, t) = 0$, $T(L, t) = 0$, and the initial condition is $T(x,0) = T_0(x/L)$, where T_0 is a constant. (Answer: $\frac{2T_0}{\pi}e^{-\pi^2 Dt/L^2}\sin(\pi x/L)$.)

(8) The thermal diffusivity *D* of the earth is about 10^{-6} m²/s. Suppose that during an ice age a certain portion of the earth is covered with ice for 10,000 years. Estimate how far into the earth the thermal effects of the ice age will penetrate. (Because the diffusion time is not defined precisely, there is a range of possible answers, roughly from 0.5 km to 2 km.)

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SEPARATION OF VARIABLES, PART I

(9) Use separation of variables to solve the initial value problem given below for a modified diffusion equation, in which β is a positive constant. The initial temperature T_0 and the boundary temperature T_1 are also constant. Don't spend a lot of time evaluating integrals, but carry the problem far enough to define the relevant coefficients in terms of integrals.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - \beta T,\ 0 < x < L,\ t > 0,$$
$$\text{with } T(0,t) = 0,\ T(L,t) = T_1, \text{ and } T(x,0) = T_0.$$

Answer:

$$T(x,t) = T_s(x) + \hat{T}(x,t), \text{ where } T_s(x) = T_1\frac{\sinh[(\beta / D)^{1/2}x]}{\sinh[(\beta / D)^{1/2}L]}$$

$$\text{and } \hat{T}(x,t) = \sum_{n=1}^{\infty} C_n e^{-(\beta + n^2\pi^2 D/L^2)t}\sin(n\pi x / L)$$

$$\text{where } C_n = \frac{2}{L}\int_0^L \left[T_0 - T_s(x)\right]\sin(n\pi x / L)dx = \frac{2T_0}{n\pi}(1-(-1)^n) + \frac{2(-1)^n n\pi T_1}{(n\pi)^2 + (\beta L^2 / D)}.$$