ME 201/MTH 281/ME 400/CHE 400 ASSIGNMENT #9 SOLUTIONS

NOV 19, 2009

(1) According to the derivative rule, $\mathcal{F}\{f'\} = ik\mathcal{F}\{f\}$.
For $f = e^{-x^2}$, $f' = -2x\,e^{-x^2}$, so

$$\mathcal{F}\{xe^{-x^2}\} = -\frac{1}{2}\mathcal{F}\{-2xe^{-x^2}\} = -\frac{ik}{2}\mathcal{F}\{e^{-x^2}\}$$
$$= -\frac{ik\sqrt{\pi}}{2}e^{-k^2/4}$$

(2) (a) We let $\hat{x} = \frac{x}{a}$, $\hat{y} = \frac{y}{a}$, $\hat{\Phi} = \frac{\Phi}{\Phi_0}$. Then

$$\frac{\partial}{\partial x} = \frac{d\hat{x}}{dx}\frac{\partial}{\partial \hat{x}} = \frac{1}{a}\frac{\partial}{\partial \hat{x}}, \quad \frac{\partial}{\partial y} = \frac{d\hat{y}}{dy}\frac{\partial}{\partial \hat{y}} = \frac{1}{a}\frac{\partial}{\partial \hat{y}},$$

so
$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{a^2}\left(\frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{y}^2}\right).$$

This gives
$$\nabla^2\Phi = \frac{\Phi_0}{a^2}\left(\frac{\partial^2\hat{\Phi}}{\partial \hat{x}^2} + \frac{\partial^2\hat{\Phi}}{\partial \hat{y}^2}\right),$$

so the equation $\nabla^2\Phi = 0$ becomes $\frac{\partial^2\hat{\Phi}}{\partial \hat{x}^2} + \frac{\partial^2\hat{\Phi}}{\partial \hat{y}^2} = 0$.

The condition $\Phi \to 0$ as $y \to \infty$ becomes $\hat{\Phi} \to 0$ as $\hat{y} \to \infty$.
The boundary condition is

$$\hat{\Phi}(\hat{x},0) = \frac{1}{\Phi_0}\Phi(x,0) = \frac{x}{a}e^{-(x/a)^2} = \hat{x}e^{-\hat{x}^2}.$$

The complete problem in the scaled variables is

$$\frac{\partial^2\hat{\Phi}}{\partial \hat{x}^2} + \frac{\partial^2\hat{\Phi}}{\partial \hat{y}^2} = 0, \quad -\infty < \hat{x} < \infty, \quad 0 < \hat{y} < \infty,$$

with $\hat{\Phi}(\hat{x},0) = \hat{x}e^{-\hat{x}^2}$ and $\hat{\Phi} \to 0$ as $\hat{y} \to \infty$.
From here on, we drop the hats on the scaled variables.

(b) Let $\tilde{\Phi}(k,y) = \int_{-\infty}^{\infty}\Phi(x,y)e^{-ikx}\,dx$. We transform the equation to get $-k^2\tilde{\Phi} + \frac{d^2\tilde{\Phi}}{dy^2} = 0$. The

general solution of this equation is
$$\tilde{\Phi} = Ae^{-|k|y} + Be^{|k|y}.$$
Because $\tilde{\Phi} \to 0$ as $y \to \infty$, B must be zero.

(2) (b) (continued). We impose the boundary condition at $y=0$ to get

$$\tilde{\Phi}(k,0) = A = \mathcal{F}\{\Phi(x,0)\} = -\frac{ik\sqrt{\pi}}{2}e^{-k^2/4}.$$

Then $\tilde{\Phi}(k,y) = -\frac{ik\sqrt{\pi}}{2}e^{-|k|y}e^{-\frac{k^2}{4}}$. We get Φ from the inversion integral:

$$\Phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}(k,y)\,e^{ikx}\,dk$$

$$= \frac{-i}{4\sqrt{\pi}}\int_{-\infty}^{\infty}k\,e^{-|k|y}e^{-\frac{k^2}{4}}e^{ikx}\,dk.$$

(c) We use the expression derived in (b) to calculate $\partial\Phi/\partial y$:

$$\frac{\partial\Phi}{\partial y}(x,y) = \frac{i}{4\sqrt{\pi}}\int_{-\infty}^{\infty}k|k|\,e^{-|k|y}e^{-\frac{k^2}{4}}e^{ikx}\,dk.$$

On the boundary $y=0$ this is

$$\frac{\partial\Phi}{\partial y}(x,0) = \frac{i}{4\sqrt{\pi}}\int_{-\infty}^{\infty}k|k|\,e^{-\frac{k^2}{4}}e^{ikx}\,dk.$$

The function $k|k|$ is odd, $e^{-\frac{k^2}{4}}$ is even, and $e^{ikx} = \underbrace{\cos kx}_{\text{even}} + i\underbrace{\sin kx}_{\text{odd}}$. The odd part of the integrand contributes nothing to the integral, so

$$\frac{\partial\Phi}{\partial y}(x,0) = \frac{i}{4\sqrt{\pi}}\int_{-\infty}^{\infty}k|k|\,e^{-\frac{k^2}{4}}\,i\sin(kx)\,dk$$

$$= -\frac{1}{2\sqrt{\pi}}\int_{0}^{\infty}k^2 e^{-\frac{k^2}{4}}\sin(kx)\,dk.$$

(d) See Mathematica notebook.

(3) We are going to use similarity methods, so we scale out as many parameters as possible. We introduce a scaled temperature $\hat{T}$ by

$$\hat{T}(x,t) = \frac{T - \frac{1}{2}(T_1+T_0)}{\frac{1}{2}(T_1-T_0)}.$$

It is easy to show that $\hat{T}$ satisfies the original equation:

$$\frac{\partial \hat{T}}{\partial t} = D\frac{\partial^2 \hat{T}}{\partial x^2}, \quad -\infty<x<\infty, \quad t>0.$$

The initial condition is now

$$\hat{T}(x,0) = \begin{cases} -1, & x<0 \\ 1, & x\geq 0 \end{cases}.$$

$\hat{T}$ is dimensionless, and therefore can depend on x, t, D only in dimensionless combinations. We note also that x, t, D are the only quantities on which $\hat{T}$ depends. As we showed in class, there is only one such dimensionless combination:

$$\eta = \frac{x}{2\sqrt{Dt}}.$$

(Of course any function of this η is also a valid dimensionless combination. We could use η^2 or $1/\eta$ for example. The 2 in the expression for η is cosmetic. Hindsight from the answer shows that it helps.)

Thus we try $\hat{T} = F(\eta)$. Then

$$\frac{\partial \hat{T}}{\partial t} = F'(\eta)\frac{\partial \eta}{\partial t} = -\frac{1}{2}F'(\eta)\frac{\eta}{t}. \quad \frac{\partial \hat{T}}{\partial x} = F'(\eta)\frac{\partial \eta}{\partial x} = \frac{F'}{2\sqrt{Dt}}.$$

and $\frac{\partial^2 \hat{T}}{\partial x^2} = \frac{F''}{4Dt}$. We substitute these into the equation to get

(3) (continued)

$$-\frac{1}{2}F'(\eta)\frac{\eta}{t} = D\frac{1}{4Dt}F''(\eta)$$

or

$$F''(\eta) + 2\eta F'(\eta) = 0.$$

This equation and the procedure leading to it are the same as in class.

The equation is separable:

$$\frac{F''}{F'} = -2\eta$$

so $\ln F' = -\eta^2 + \text{constant}$

$$F' = Ce^{-\eta^2}.$$

We integrate once more to get

$$F(\eta) = C\int_0^{\eta} e^{-\eta'^2}\,d\eta' + E.$$

Now we impose the initial condition. For $x>0$, $\hat{T} \xrightarrow[t\to 0]{} 1$. As $t\to 0$ for $x>0$, $\eta\to\infty$, so

$$1 = C\int_0^{\infty} e^{-\eta'^2}\,d\eta' + E = C\frac{\sqrt{\pi}}{2} + E.$$

As $t\to 0$ for $x<0$, $\eta\to-\infty$, and $\hat{T}\to -1$, so

$$-1 = C\int_0^{-\infty} e^{-\eta'^2}\,d\eta' + E = -C\frac{\sqrt{\pi}}{2} + E.$$

We solve these equations for C and E to get $E=0$, $C=\frac{2}{\sqrt{\pi}}$, so

$$\hat{T} = F(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta'^2}\,d\eta' = \operatorname{erf}(\eta).$$

Reverting to the original T, we get

$$T(x,t) = \frac{1}{2}(T_0+T_1) + \frac{1}{2}(T_1-T_0)\operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right).$$

(4) We choose the fundamental dimensions to be ~~four~~ mass M, length L, and time T. In this system a force has dimensions MLT^{-2}, and energy has dimensions of force times length, hence ML^2T^{-2}. The relevant quantities are the radius R, which we seek, ~~and~~ the initial density ρ with $[\rho] = ML^{-3}$, the energy E, $[E] = ML^2T^{-2}$, and the time t, $[t] = T$. A little experimenting shows that the quantity $\left[\frac{Et^2}{\rho}\right]^{\frac{1}{5}}$ has the dimensions of length. Then the quantity

$$\frac{R}{\left[\frac{Et^2}{\rho}\right]^{\frac{1}{5}}}$$

is dimensionless. It depends only on E, ρ and t, and, because it is dimensionless, can only depend on a dimensionless combination of E, ρ and t. There are no dimensionless combinations of E, ρ and t, so the above combination must be equal to a dimensionless constant k. Then

$$R(t) = k\left[\frac{Et^2}{\rho}\right]^{\frac{1}{5}}$$

so $R(t)$ grows like $t^{2/5}$.

(5) (a) $y = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$

$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$

$xy' = a_1x + 2a_2x^2 + 3a_3x^3 \cdots = \sum_{n=0}^{\infty} n a_n x^n$

$y'' = 2a_2 + (2)(3)a_3x + (3)(4)a_4x^2 \cdots = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$

The recurrence relation is

$$a_n + n a_n + (n+1)(n+2)a_{n+2} = 0$$

so $a_{n+2} = -\dfrac{a_n}{n+2}$

(5) (a) (continued) Because $y(0)=1$, $a_0=1$. Because $y'(0)=0$, $a_1=0$. Then only the even terms are nonzero.

$$a_0 = 1$$
$$a_2 = -\frac{a_0}{2} = -\frac{1}{2}$$
$$a_4 = -\frac{a_2}{4} = \frac{1}{8} = \frac{1}{2^2 \cdot 2!}$$
$$a_6 = -\frac{a_4}{6} = -\frac{1}{48} = \frac{1}{2^3 \cdot 3!}$$
$$a_8 = -\frac{a_6}{8} = \frac{1}{384} = \frac{1}{2^4 \cdot 4!}$$
$$\vdots$$
$$a_{2k} = \frac{(-1)^k}{2^k \cdot k!}$$

The first four nonzero terms are $y = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6$.
The series is
$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \cdot k!} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{x^2}{2}\right)^k$$

We recognize the series as the series for the exponential function, so
$$y(x) = e^{-\frac{x^2}{2}}.$$

(b) $y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$

$$y'' = 2a_2 + (2)(3)a_3 x + (3)(4)a_4 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)(n+2) \cdot a_{n+2} x^n$$

The recurrence relation is then
$$(n+1)(n+2)a_{n+2} - 4a_n = 0$$
so
$$a_{n+2} = \frac{4a_n}{(n+1)(n+2)}$$

From the initial conditions we get $a_0 = 0$ and $a_1 = 2$. Then all of the even coefficients are zero, and

(5) (b) (continued)

$$a_1 = 2, \quad a_3 = \frac{4a_1}{(2)(3)} = \frac{2\cdot 2^2}{3!} = \frac{2^3}{3!}$$

$$a_5 = \frac{4a_3}{(4)(5)} = \frac{2^5}{5!}, \quad a_7 = \frac{4a_5}{(6)(7)} = \frac{2^7}{7!}$$

The first four nonzero terms are $y = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{315}x^7$

The series is $\displaystyle\sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!}x^{2k+1} = \sum_{k=0}^{\infty}\frac{(2x)^{2k+1}}{(2k+1)!}$

$$= \sinh(2x).$$

(c) We introduce $\eta = x-2$. Then $\frac{d}{dx} = \frac{d}{d\eta}$, $x = (\eta+2)$, and $(x-1) = \eta+1$. The equation in terms of η is

$$(\eta+2)\frac{d^2y}{d\eta^2} + (\eta+1)\frac{dy}{d\eta} - 2y = 0$$

$$-2y = -2(a_0 + a_1\eta + a_2\eta^2 \cdots) = -2\sum_{n=0}^{\infty} a_n\eta^n$$

$$\frac{dy}{d\eta} = a_1 + 2a_2\eta + 3a_3\eta^2 + \cdots = \sum_{n=0}^{\infty}(n+1)a_{n+1}\eta^n$$

$$\eta\frac{dy}{d\eta} = a_1\eta + 2a_2\eta^2 + \cdots = \sum_{n=0}^{\infty} n a_n\eta^n$$

$$2\frac{d^2y}{d\eta^2} = 2(2a_2 + (2)(3)a_3\eta + (3)(4)a_4\eta^2)\cdots$$
$$= \sum_{n=0}^{\infty} 2(n+1)(n+2)a_{n+2}\eta^n$$

$$\eta\frac{d^2y}{d\eta^2} = 2a_2\eta + (2)(3)a_3\eta^2 + \cdots = \sum_{n=0}^{\infty} n(n+1)a_{n+1}\eta^n.$$

The recurrence relation is

$$n(n+1)a_{n+1} + 2(n+1)(n+2)a_{n+2} + na_n + (n+1)a_{n+1} - 2a_n = 0$$

(5) (c) (continued) We solve for a_{n+2}:

$$a_{n+2} = -\frac{(n+1)\,a_{n+1}}{2(n+2)} - \frac{(n-2)\,a_n}{2(n+1)(n+2)}.$$

Then
$$a_2 = -\frac{(1)\,a_1}{2(2)} - \frac{(-2)\,a_0}{2(2)}$$
$$= -\frac{a_1}{4} + \frac{a_0}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$a_3 = -\frac{(2)\,a_2}{2(3)} - \frac{(-1)\,a_1}{2(2)(3)}$$
$$= -\frac{a_2}{3} + \frac{a_1}{12} = -\frac{1}{4} - \frac{1}{12} = -\frac{1}{3}$$

The first four nonzero terms are
$$y = 1 - \eta + \frac{3}{4}\eta^2 - \frac{1}{3}\eta^3 \quad \text{where } \eta = x-2.$$

CHALLENGE PROBLEM

(a) We start by Fourier transforming the equation. Let
$$\tilde{T}(k,t) = \int_{-\infty}^{\infty} T(x,t)\, e^{-ikx}\, dx$$

Then
$$\frac{d\tilde{T}}{dt} = -Dk^2\tilde{T}.$$

We integrate this to get
$$\tilde{T}(k,t) = \tilde{T}(k,0)\, e^{-Dk^2 t}.$$

By transforming $T(x,0)$, we get $\tilde{T}(k,0)$:
$$\tilde{T}(k,0) = \int_{-\infty}^{\infty} T(x,0)\, e^{-ikx}\, dx = \frac{E_0}{\rho c}\int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = \frac{E_0}{\rho c}.$$

Then
$$\tilde{T}(k,t) = \frac{E_0}{\rho c}\, e^{-Dk^2 t}.$$

(b) The relevant entry in the table is

$f(x)$	$\tilde{f}(k)$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}\, e^{-\frac{k^2}{4a}}$.

(a) CHALLENGE PROBLEM (continued)

(b) Then the inverse of $e^{-\frac{k^2}{4a}}$ is $\sqrt{\frac{a}{\pi}}\, e^{-ax^2}$.
In our case $Dt = \frac{1}{4a}$, so $a = \frac{1}{4Dt}$, and the inverse of $\frac{E_0}{\rho c} e^{-Dk^2 t}$

$$\text{is } T(x,t) = \frac{E_0}{\rho c}\sqrt{\frac{a}{\pi}}\, e^{-ax^2}$$
$$= \frac{E_0}{\rho c}\frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}$$
$$= \frac{E_0}{2\rho c}\frac{1}{\sqrt{\pi D t}}\, e^{-\frac{x^2}{4Dt}}$$

(c) $$E(x_1, x_2, t) = \int_{x_1}^{x_2} \rho c T(x,t)\, dx = \frac{E_0}{2\sqrt{\pi D t}} \int_{x_1}^{x_2} e^{-\frac{x^2}{4Dt}}\, dx.$$

In the integral, let $z = \frac{x}{\sqrt{4Dt}}$. Then

$$E(x_1, x_2, t) = \frac{E_0}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-z^2}\, dz, \quad z_1 = \frac{x_1}{2\sqrt{Dt}}, \quad z_2 = \frac{x_2}{2\sqrt{Dt}}.$$

so $$E(x_1, x_2, t) = \frac{1}{2}E_0\left\{\frac{2}{\sqrt{\pi}}\int_0^{z_2} e^{-z^2}dz - \frac{2}{\sqrt{\pi}}\int_0^{z_1} e^{-z^2} dz\right\}$$
$$= \frac{1}{2}E_0\left\{\operatorname{erf}\left(\frac{x_2}{2\sqrt{Dt}}\right) - \operatorname{erf}\left(\frac{x_1}{2\sqrt{Dt}}\right)\right\}$$

(d)
For $x_1 = -x_2$, $\operatorname{erf}\left(\frac{x_1}{2\sqrt{Dt}}\right) = \operatorname{erf}\left(\frac{-x_2}{2\sqrt{Dt}}\right) = -\operatorname{erf}\left(\frac{x_2}{2\sqrt{Dt}}\right)$

So $E(-x_2, x_2, t) = E_0 \operatorname{erf}\left(\frac{x_2}{2\sqrt{Dt}}\right)$.

If the slab contains half the energy, then

$$E_0 \operatorname{erf}\left(\frac{x_2}{2\sqrt{Dt}}\right) = \frac{1}{2}E_0, \text{ so}$$
$$\operatorname{erf}\left(\frac{x_2}{2\sqrt{Dt}}\right) = \frac{1}{2}.$$

CHALLENGE PROBLEM

(d) (continued). From tables of the error function or Mathematica, one finds that erf $\simeq \frac{1}{2}$ when its argument is $\frac{1}{2}$. (More precisely erf$(\frac{1}{2}) = 0.5205$, which is close enough to $\frac{1}{2}$) Then

$$\frac{x_2}{2\sqrt{Dt}} = \frac{1}{2}$$

$$\text{so} \quad x_2(t) = \sqrt{Dt}.$$

Note that this is the usual scale relation for diffusion: $t = \frac{x_2^2}{D}$.

(e) The total energy at time t is

$$\lim_{\substack{x_2 \to \infty \\ x_1 \to -\infty}} E(x_1, x_2, t) = \frac{E_0}{2}\{\text{erf}(\infty) - \text{erf}(-\infty)\}$$

$$= \frac{E_0}{2}\{1 - (-1)\} = E_0.$$

The Fourier transform of T is

$$\mathcal{T}(k,t) = \int_{-\infty}^{\infty} T(x,t)\, e^{-ikx}\, dx = \frac{E_0}{\rho c} e^{-Dk^2 t}$$

Then $\tilde{T}(0,t) = \int_{-\infty}^{\infty} T(x,t)\, dx = \frac{E_0}{\rho c}$,

So the total energy is $\int_{-\infty}^{\infty} \rho c T(x,t)\, dx = \rho c \frac{E_0}{\rho c}$

$$= E_0.$$

ME201/MTH281/ME400/CHE400 Assignment #9 Solutions Problem 2(d)

Problem 2

(d)

We let f(x) stand for $\partial\Phi/\partial y|_{y=0}$. As shown on the solution sheet, f is given by

```
f[x_] := -(1/(2 Sqrt[π])) NIntegrate[k^2 Exp[-k^2/4] Sin[k x], {k, 0, ∞}]
```

We try a few values.

```
f[0.5]
```

```
-0.803652
```

```
f[1.0]
```

```
-0.521221
```

```
f[2.0]
```

NIntegrate::deodiv : DoubleExponentialOscillatory returns a finite integral estimate, but the integral might be divergent. ≫

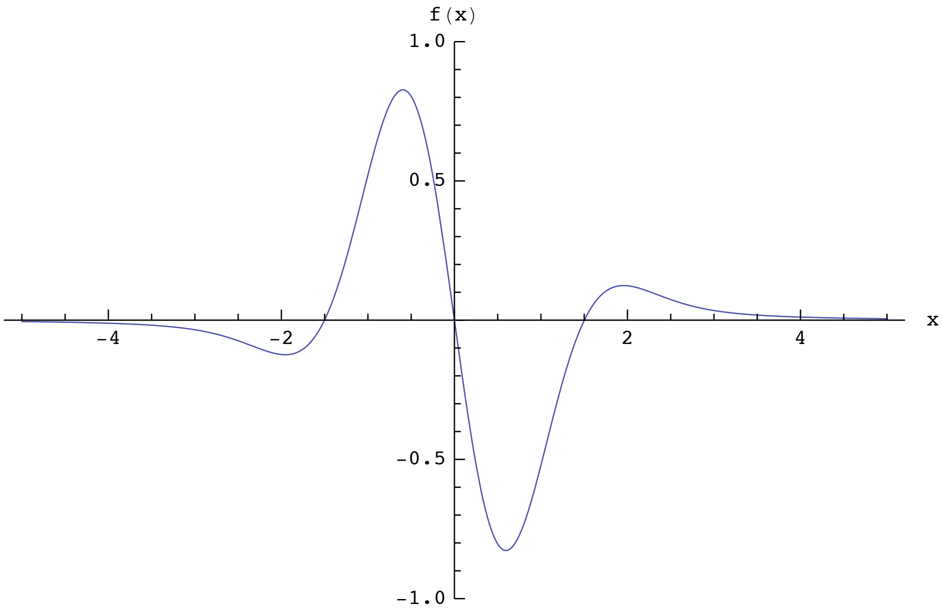
```
0.123425
```

```
f[3.0]
```

```
0.0345369
```

The message shows that *Mathematica* is struggling a bit. We continue with the plot.

```
Plot[f[x], {x, -5, 5}, PlotRange → {-1, 1}, AxesLabel → {"x", "f(x)"}]
```



We estimate the maximum and minimum values as about ±0.82.

We did numerically the integral defining f(x). Let's see if by any chance *Mathematica* can do this integral analytically.

$$\text{fanalyt[x\_]} = -\left(1 \Big/ \left(2\sqrt{\pi}\right)\right) \int_0^{\infty} k^2\, \text{Exp}\left[-\frac{k^2}{4}\right] \text{Sin[k x]}\, dk$$

$$-\frac{2\left(x + \text{DawsonF}[x] - 2\, x^2\, \text{DawsonF}[x]\right)}{\sqrt{\pi}}$$

The good news is that *Mathematica* has returned an analytical result. The bad news is that the answer has been expressed in terms of something called the DawsonF function. We go to Help to see what this is. Here is what we find in Help:

The Dawson integral is defined by $F(x) = \exp\left(-x^2\right) \int_0^x \exp\left(y^2\right) dy$.

Interesting. Let's see if our analytical result gives the same answers as the numerical integration.

```
fanalyt[0.5]
```

-0.803652

```
fanalyt[1.]
```

-0.521221