ME201/MTH281/ME400/CHE400 ASSIGNMENT # 8 SOLUTIONS

NOV 5, 2009

(1) We try $y = \cos(\omega t)\,\phi(x)$. We substitute this into the equation to get $-\omega^2\phi = c^2\phi''$, or

$$\phi'' + k^2\phi = 0, \quad k^2 = \omega^2/c^2.$$

The general solution is $\phi = A\cos kx + B\sin kx$. From $\phi(0)=0$ we get $A=0$. From $\phi'(L)=0$ we get $\cos kL = 0 \Rightarrow kL = (n-\frac{1}{2})\pi$, $n=1,2,3\ldots$ Then $\omega_n = ck_n = (n-\frac{1}{2})\pi c/L$, and the linear frequencies are

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{(n-\frac{1}{2})c}{2L}.$$

The fundamental is $\nu_1 = \frac{c}{4L}$. The set of frequencies is $\{\nu_1, 3\nu_1, 5\nu_1, 7\nu_1 \ldots\}$.
In this case, only the odd multiples of the fundamental are present. (For the string fixed at both ends, the set of frequencies includes all integer multiples of the fundamental.)

(2) Let x be the coordinate along the axis of the organ pipe. Then as we showed in class, the pressure variations p in the pipe satisfy

$$\frac{\partial^2 p}{\partial t^2} = c^2\,\frac{\partial^2 p}{\partial x^2}$$

where c is the sound speed in the air. The boundary condition at an open end is $p=0$ (approximately), so our modes must satisfy $p=0$ at $x=0$ and $x=L$. We try $p = \cos(\omega t)\,\phi(x)$ to get $-\omega^2\phi = c^2\phi''$, or $\phi'' + k^2\phi = 0$, where $k = \omega/c$. The general solution is $\phi = A\cos kx + B\sin kx$, and $\phi(0)=0 \Rightarrow A=0$, $\phi(L)=0 \Rightarrow \sin kL = 0 \Rightarrow kL = n\pi$, $n=1,2,\ldots$

Then the frequencies are $\omega_n = ck_n = \frac{n\pi c}{L}$, and the

(2) (continued) linear frequencies are $\nu_n = \frac{nc}{2L}$.
The fundamental is $\nu_1 = \frac{c}{2L}$.

To finish the problem we need the speed of sound in air, which can be looked up in many references. From Fundamentals of Physical Acoustics, David Blackstock, John Wiley 2000, p. 512, we get $c = 343.0$ m/s at 20 °C. Then

$$L = \frac{c}{2\nu_1} = \frac{(343.0)}{(2)(65.41)} = 2.622 \text{ m}.$$

To find the sound speed at 5°C, we use the fact that the sound speed is proportional to $\sqrt{T}$, where T is the absolute temperature. Then

$$c_5 = c_{20}\sqrt{\frac{T_5}{T_{20}}} = (343.0)\sqrt{\frac{273.15+5}{273.15+20}}$$

$$= 334.1 \text{ m/s}.$$

This will give a frequency of $\frac{c}{2L} = \frac{334.1}{(2)(2.622)} = 63.71$ Hz. The next note down from C at 65.41 Hz is B at 61.74 Hz, so the drop in pitch would certainly be noticeable. However the pitch of other organ pipes is changed also, and it turns out that the frequency ratio for two different pipes is unchanged:

$$\frac{\nu_{pipe1}}{\nu_{pipe2}} = \frac{c/2L_1}{c/2L_2} = \frac{L_2}{L_1}$$

which is independent of c.

(3) (a) We try $\psi(x,y,t) = \cos(\omega t)\,\phi(x,y)$. We substitute this into the equation to get

$$c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) - \frac{\alpha}{\sigma}\phi = -\omega^2\phi.$$

(3) (a) (continued) We may write this as

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\beta^2\phi=0, \text{ where } \beta^2=\frac{\omega^2}{c^2}-\frac{\alpha}{\sigma c^2}.$$

The boundary condition is that $\phi=0$ on the edge of the membrane. This equation is known as the Helmholtz equation. We solve it by separation of variables. We try $\phi=F(x)G(y)$. We substitute this into the equation to get

$$\frac{1}{F}\frac{d^2F}{dx^2}+\frac{1}{G}\frac{d^2G}{dy^2}+\beta^2=0.$$

The separation has worked. We have $\frac{1}{F}\frac{d^2F}{dx^2}=-\lambda_x$ and $\frac{1}{G}\frac{d^2G}{dy^2}=-\lambda_y$, with $\lambda_x+\lambda_y=\beta^2$. Then

$$\frac{d^2F}{dx^2}+\lambda_x F=0, \quad 0<x<a, \quad F(0)=0, \quad F(a)=0$$

and

$$\frac{d^2G}{dy^2}+\lambda_y G=0, \quad 0<y<b, \quad G(0)=0, \quad G(b)=0$$

We have solved this problem many times. The result is

$$F_m(x)=\sin(m\pi x/a), \quad \lambda_{x,m}=\frac{m^2\pi^2}{a^2}, \quad m=1,2,\ldots$$

$$F_n(y)=\sin(n\pi y/b), \quad \lambda_{y,n}=\frac{n^2\pi^2}{b^2}, \quad n=1,2,\ldots$$

Then $\beta^2=\lambda_x+\lambda_y=\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}$, so

$$\omega_{mn}^2=\frac{\alpha}{\sigma}+c^2\beta^2=\frac{\alpha}{\sigma}+c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right).$$

The frequencies in Hz are given by

$$\nu_{mn}^2=\frac{\omega_{mn}^2}{4\pi^2}=\nu_0^2+\frac{c^2}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right), \quad m,n=1,2,3\ldots$$

where $\nu_0^2=\frac{\alpha}{4\sigma\pi^2}$.

(3)(a) (continued) The term $\frac{c^2}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$

increases without limit as m, n increase. Eventually it will be much larger than ν_0^2, and then

$$\nu_{mn}^2 \simeq \frac{c^2}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

and these are the frequencies when $\alpha=0$

(b) In the lattice space (m,n), every point with positive integer coordinates corresponds to a mode. Each mode is associated with unit area in that space.

The number of modes with frequencies $\leq \nu$ is the number of points (m,n) with integer coordinates satisfying

$$\nu_{mn}^2 \leq \nu^2$$

or

$$\nu_0^2 + \frac{c^2}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right) \leq \nu^2$$

or

$$\left(\frac{m}{\frac{2a}{c}\sqrt{\nu^2-\nu_0^2}}\right)^2 + \left(\frac{n}{\frac{2b}{c}\sqrt{\nu^2-\nu_0^2}}\right)^2 \leq 1.$$

The boundary of the above region is ~~......~~ a part of an ellipse in the first quadrant. The ellipse has semi-axes $a_m = \frac{2a}{c}\sqrt{\nu^2-\nu_0^2}$, $a_n = \frac{2b}{c}\sqrt{\nu^2-\nu_0^2}$.

The area of the ellipse is $\pi a_m a_n$, and the area in the first quadrant is $\frac{1}{4}\pi a_m a_n$. Thus if $N(\nu)$ is the number of modes with frequencies $\leq \nu$, we have $N(\nu) = \frac{1}{4}\pi\, \frac{2a}{c}\sqrt{\nu^2-\nu_0^2}\cdot \frac{2b}{c}\sqrt{\nu^2-\nu_0^2} = \frac{ab\pi(\nu^2-\nu_0^2)}{c^2}$.

(3) (b) (continued) This equation is valid only if it gives a large number because we have not accurately tracked modes very near the bounding ellipse (some may be just outside and others just inside).

For the parameters given, $C^2 = \frac{T}{\sigma} = \frac{100}{0.1} = 1000\ \frac{m^2}{s^2}$, and $\nu_0 = \sqrt{\frac{\alpha}{\sigma}} \cdot \frac{1}{2\pi} = \sqrt{\frac{10^3}{0.1}}\ \frac{1}{2\pi} = 15.91$ Hz.

Then for $\nu_1 = 2000$, $\nu_2 = 4000$, the number we want is

$$N(\nu_2) - N(\nu_1) = \frac{ab\pi}{C^2}(\nu_2^2 - \nu_0^2) - \frac{ab\pi}{C^2}(\nu_1^2 - \nu_0^2)$$

$$= \frac{ab\pi}{C^2}(\nu_2^2 - \nu_1^2) = \frac{(0.4)(0.4)(\pi)}{1000}(4000^2 - 2000^2)$$

$$= 6032 \text{ modes}$$

(4) (a) $$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{b} e^{(a-ik)x} dx = \frac{e^{(a-ik)b}}{a-ik}$$

(b) $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{b} e^{ax} dx = \frac{1}{a} e^{ab}, \quad \tilde{f}(0) = \frac{1}{a} e^{ab}.$$

(c) Parseval's theorem says that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk.$$

The integral on the left is $\int_{-\infty}^{b} |e^{ax}|^2 dx = \int_{-\infty}^{b} e^{2ax} dx = \frac{e^{2ab}}{2a}$.

$\tilde{f}(k) = \dfrac{e^{ab} e^{-ikb}}{a-ik}$, so $|\tilde{f}(k)|^2 = \dfrac{e^{2ab}}{a^2+k^2}$, so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk = \frac{e^{2ab}}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{a^2+k^2} = \frac{e^{2ab}}{\pi}\int_{0}^{\infty} \frac{dk}{a^2+k^2}$$

$$= \frac{e^{2ab}}{\pi} \cdot \frac{\pi}{2a} = \frac{e^{2ab}}{2a}. \quad \text{It checks.}$$

(5) We take the Fourier transform of the equation:

$$\mathcal{F}\{y''\} = -k^2\tilde{y}, \quad \mathcal{F}\{-y\} = -\tilde{y}, \text{ and}$$

$$\mathcal{F}\{e^{-\alpha|x|}\} = \frac{2\alpha}{\alpha^2+k^2} \quad \text{(given in class)}$$

Then

$$-k^2\tilde{y} - \tilde{y} = \frac{2\alpha}{\alpha^2+k^2}$$

$$\text{so } \tilde{y} = -\frac{2\alpha}{(\alpha^2+k^2)(1+k^2)}$$

We use partial fractions to transform the right-hand side:

$$\frac{1}{\alpha^2+k^2}\,\frac{1}{1+k^2} = -\frac{1}{\alpha^2-1}\,\frac{1}{\alpha^2+k^2} + \frac{1}{\alpha^2-1}\,\frac{1}{k^2+1}$$

Then

$$\tilde{y} = \frac{2\alpha}{\alpha^2-1}\,\frac{1}{\alpha^2+k^2} - \frac{2\alpha}{\alpha^2-1}\,\frac{1}{1+k^2}.$$

From the table given in class, we have

$$\mathcal{F}^{-1}\left\{\frac{1}{\beta^2+k^2}\right\} = \frac{1}{2\beta}e^{-\beta|x|}, \quad \beta>0$$

Then

$$y(x) = \frac{2\alpha}{\alpha^2-1}\left\{\mathcal{F}^{-1}\left(\frac{1}{\alpha^2+k^2}\right) - \mathcal{F}^{-1}\left(\frac{1}{1+k^2}\right)\right\}$$

$$= \frac{2\alpha}{\alpha^2-1}\left(\frac{1}{2\alpha}e^{-\alpha|x|} - \frac{1}{2}e^{-|x|}\right)$$

$$= \frac{1}{\alpha^2-1}\left(e^{-\alpha|x|} - \alpha e^{-|x|}\right).$$

<u>CHALLENGE PROBLEM</u>

The equation and boundary conditions are

$$\sigma\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \quad 0<x<L,\ t>0,$$

$$y(0,t) = 0 \quad \text{(fixed left end)}$$

and $m\dfrac{\partial^2 y}{\partial t^2}(L,t) = -T\dfrac{\partial y}{\partial x}(L,t)$ (frictionless slider of mass m on the right)

CHALLENGE PROBLEM (continued)

(a) We look for normal modes of the form $y(x,t) = \cos\omega t\, \phi(x)$. The equation and boundary conditions become

$$\frac{d^2\phi}{dx^2} + \frac{\sigma\omega^2}{T}\phi = 0, \quad 0<x<L$$

$$\phi(0) = 0, \quad T\phi'(L) = m\omega^2\phi(L).$$

Our task is to determine the values of ω^2 for which there are non-trivial solutions. This looks a lot like a regular Sturm-Liouville system, but is not because the eigenvalue ω^2 also appears in the boundary condition.

(b) Let $k^2 = \frac{\sigma\omega^2}{T}$. Then the general solution of the equation is $A\cos kx + B\sin kx$. Because $\phi(0)=0$, we must have $A=0$. Then we can take $\phi = \sin kx$. The boundary condition at $x=L$ becomes

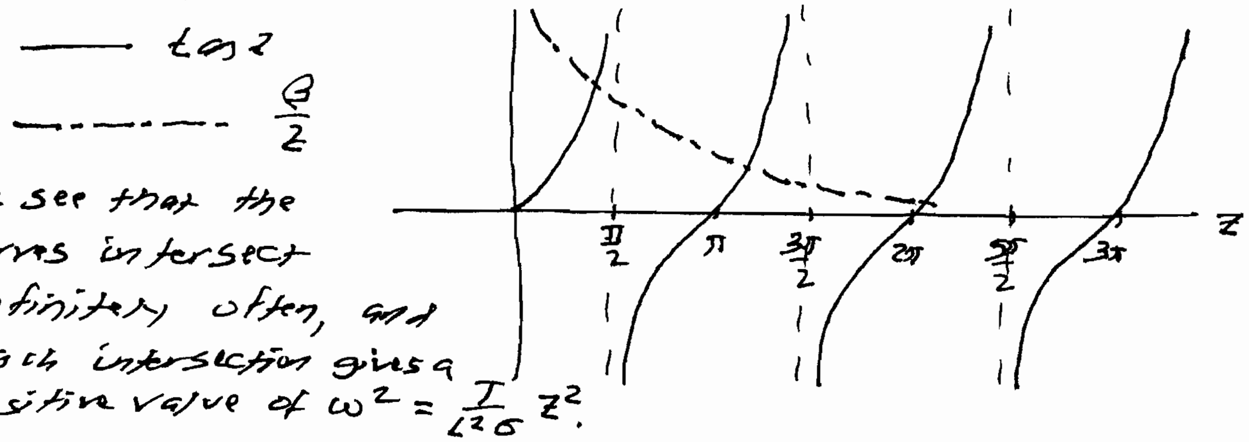
$$Tk\cos kL = m\omega^2 \sin kL.$$

We let $z = kL$, so $\omega^2 = \frac{Tk^2}{\sigma}$. Then

$$Tk\cos z = \frac{mTk^2}{\sigma}\sin z$$

$$\text{so} \quad \tan z = \frac{\sigma}{km} = \frac{\beta}{z}, \quad \text{where } \beta = \frac{\sigma L}{m}.$$

Note that β is the total mass of the string divided by the mass of the slider.



We see that the curves intersect infinitely often, and each intersection gives a positive value of $\omega^2 = \frac{T}{L^2\sigma}z^2$.

CHALLENGE PROBLEM (5) (continued.

Now we show that there are no negative value of ω^2 (such value would correspond to growing or decaying non-oscillatory motions).

We have $\frac{d^2\phi}{dx^2} + \frac{\sigma\omega^2}{T}\phi = 0, \quad 0 < x < L,$

and $\phi(0) = 0$, and $\phi'(L) = \frac{m\omega^2}{T}\phi(L)$.

Multiply the equation by ϕ to get

$$\phi''\phi + \frac{\sigma\omega^2}{T}\phi^2 = 0$$

or $$(\phi'\phi)' - \phi'^2 + \frac{\sigma\omega^2}{T}\phi^2 = 0$$

Integrate from $x=0$ to $x=L$.

$$\phi'\phi\Big|_0^L - \int_0^L \phi'^2\,dx + \frac{\sigma\omega^2}{T}\int_0^L \phi^2\,dx = 0$$

$$\phi'(L)\phi(L) - \int_0^L \phi'^2\,dx + \frac{\sigma\omega^2}{T}\int_0^L \phi^2\,dx = 0$$

$$\frac{m\omega^2}{T}[\phi(L)]^2 - \int_0^L \phi'^2\,dx + \frac{\sigma\omega^2}{T}\int_0^L \phi^2\,dx = 0$$

so $$\omega^2 = \frac{\int_0^L \phi'^2\,dx}{\frac{m}{T}[\phi(L)]^2 + \frac{\sigma}{T}\int_0^L \phi^2\,dx} \geq 0.$$

Therefore ω^2 is never negative. We can also rule out $\omega^2 = 0$. If $\omega^2 = 0$, then $\phi' \equiv 0$, so $\phi =$ constant, but $\phi(0) = 0$ so $\phi \equiv 0$.

(c) See Mathematica notebook.

ME201/MTH281/ME400/CHE400 Assignment #8 Solutions Challenge Problem

Challenge Problem (c)

We will use FindRoot here to find the first three frequencies of our string-slider system. We begin by specifying the parameter values.

```
L = 0.5 (** string length in m **);
σ = 4 * 10^-3 (** string density in kg/m **);
T = 40.0 (** string tension in N **);
m = 2 * 10^-3 (** slider mass in kg **);
β := (σ L / m) (** ratio of string mass to slider mass **);
```

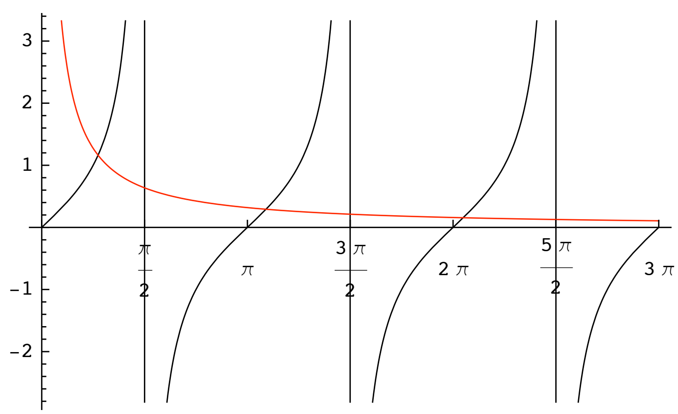
The value of β in the present case is

```
β
```

```
1.
```

As we showed on the handwritten solution sheet, the eigenvalue equation has the form tan(z) = β/z. We repeat the graph given on the solution sheet.

```
Plot[{Tan[z], β / z}, {z, 0, 3 π},
 PlotStyle → {RGBColor[0, 0, 0], RGBColor[1, 0, 0]},
 Ticks → {{0, Pi / 2, Pi, 3 Pi / 2, 2 Pi, 5 Pi / 2, 3 Pi}}, ImageSize → 250]
```



We see that there is a root in each interval of length π. A good initial guess for the nth root would be (n-1)π +

0.2. Now we define a function which returns the nth value of z. We call it zval.

```
zval[n_] := z /. FindRoot[Tan[z] == β / z, {z, (n - 1) π + 0.2}]
```

We try this out.

```
zval[1]
```

```
0.860334
```

We see that this is in the first interval (0 to $\pi/2$). We check to see that it satisfies the equation.

```
Tan[zval[1]]
```

```
1.16234
```

```
β / zval[1]
```

```
1.16234
```

Looks good. Now we define a function which gives the frequency in Hz for the nth mode. As we showed earlier, $\omega = \sqrt{(T/\sigma)}\,(z/L)$, and the frequency in Hz, which we call freq, is $\omega/(2\pi)$. Then we define

$$\text{freq[n\_]} := \frac{\sqrt{(T/\sigma)}}{2\,\pi\,L}\,\text{zval[n]}$$

The first three frequencies (in Hz) are

```
freq[1]
```

```
27.3853
```

```
freq[2]
```

```
109.041
```

```
freq[3]
```

```
204.906
```