(1) (a) Let $\hat{x} = 1 + \frac{x}{L}$. Then $\frac{\partial}{\partial x} = \frac{d\hat{x}}{dx}\frac{\partial}{\partial \hat{x}} = \frac{1}{L}\frac{\partial}{\partial \hat{x}}$.

Let $\hat{t} = \frac{D_0 t}{L^2}$. Then $\frac{\partial}{\partial t} = \frac{d\hat{t}}{dt}\frac{\partial}{\partial \hat{t}} = \frac{D_0}{L^2}\frac{\partial}{\partial \hat{t}}$.

We have $\rho = \rho_0$, $C = C_0(1+\frac{x}{L}) = C_0\hat{x}$ and $K = \frac{K_0}{1+\frac{x}{L}} = \frac{K_0}{\hat{x}}$.

Substituting these into the equation, and introducing $\hat{T} = T/T_0$, gives

$$\hat{x}\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{x}}\left(\frac{1}{\hat{x}}\frac{\partial \hat{T}}{\partial \hat{x}}\right),\quad 1<\hat{x}<2,\ \hat{t}>0.$$

with $\hat{T}(1,\hat{t})=0$, $\hat{T}(2,\hat{t})=0$, and $\hat{T}(\hat{x},0)=1$.

(b) We drop the hats. We try $T = G(t)F(x)$. This gives

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{xF}\frac{d}{dx}\left(\frac{1}{x}\frac{dF}{dx}\right) = -\lambda.$$

Then $G = \text{const}\, e^{-\lambda t}$, and F is a solution of

$$\frac{d}{dx}\left(\frac{1}{x}\frac{dF}{dx}\right) = -\lambda x F,\quad 1<x<2,\ F(1)=0,\ F(2)=0.$$

We are asked to verify that $C_1\cos\left[\frac{\sqrt{\lambda}x^2}{2}\right] + C_2\sin\left[\frac{\sqrt{\lambda}x^2}{2}\right]$ is the general solution. We check the cos solution.

$$\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\cos\left[\frac{\sqrt{\lambda}x^2}{2}\right]\right] = \frac{d}{dx}\left[\frac{1}{x}(-\sqrt{\lambda}x)\sin\left[\frac{\sqrt{\lambda}x^2}{2}\right]\right] = -\sqrt{\lambda}\frac{d}{dx}\sin\left[\frac{\sqrt{\lambda}x^2}{2}\right]$$

$= -\lambda x\cos\left[\frac{\sqrt{\lambda}x^2}{2}\right]$. A similar calculation verifies that the sin is also a solution. To solve the equation without knowing a priori the form of the solution, let $u = x^2/2$. The equation becomes $d^2F/du^2 = -\lambda F$ which is easily solved.

(c) We impose the boundary conditions:

$$F(1) = 0 = C_1\cos(\sqrt{\lambda}/2) + C_2\sin(\sqrt{\lambda}/2)$$
$$F(2) = 0 = C_1\cos(2\sqrt{\lambda}) + C_2\sin(2\sqrt{\lambda})$$

(1) (c) (continued) For a non-trivial solution, at least one of C_1, C_2 must be non-zero. $\therefore$ the determinant must vanish, so

$$\cos(\sqrt{\lambda}/2)\sin(2\sqrt{\lambda}) - \cos(2\sqrt{\lambda})\sin(\sqrt{\lambda}/2) = 0$$

$$\sin\left(2\sqrt{\lambda} - \frac{\sqrt{\lambda}}{2}\right) = 0$$

$$\sin\left(\frac{3\sqrt{\lambda}}{2}\right) = 0.$$

The zeros of the sine are $n\pi$, so

$$\frac{3\sqrt{\lambda}}{2} = n\pi \Rightarrow \lambda = \frac{4n^2\pi^2}{9}, \quad n = 1, 2, 3.$$

Before carrying out this calculation, we could have shown that all of the eigenvalues are positive, by multiplying the equation for F by F and integrating over $[1, 2]$. We do that now.

$$\frac{d}{dx}\left(\frac{1}{x}\frac{dF}{dx}\right) = -\lambda x F$$

$$F\left(\frac{1}{x}F'\right)' = -\lambda x F^2$$

$$\left(\frac{1}{x}FF'\right)' - \frac{F'^2}{x} = -\lambda x F^2$$

$$\underbrace{\int_1^2 \left(\frac{1}{x}FF'\right)' dx}_{=0 \text{ because of BC}} - \int_1^2 \frac{F'^2}{x}\,dx = -\lambda \int_1^2 x F^2\,dx$$

$$\text{so} \quad \lambda = \frac{\int_1^2 \frac{F'^2}{x}\,dx}{\int_1^2 x F^2\,dx}.$$

This shows that $\lambda \geq 0$. To have $\lambda = 0$, we must have $\int_1^2 \frac{F'^2}{x}\,dx = 0$ which requires $F' \equiv 0$. Then $F = \text{constant}$, but $F(1) = 0 \Rightarrow \text{constant} = 0$, so only the trivial solution for $\lambda = 0$. $\therefore \lambda > 0$.

(1) (c) (continued) We have $F(x) = C_1 \cos(\sqrt{\lambda}\frac{x^2}{2}) + C_2 \sin(\sqrt{\lambda}\frac{x^2}{2})$

and $C_1 \cos(\sqrt{\lambda}/2) + C_2 \sin(\sqrt{\lambda}/2) = 0$. From this last equation we get

$$C_1 = -\alpha \sin(\sqrt{\lambda}/2), \quad C_2 = \alpha \cos(\sqrt{\lambda}/2),$$

where α is arbitrary. Then

$$F(x) = \alpha\{-\sin(\sqrt{\lambda}/2)\cos(\sqrt{\lambda}x^2/2) + \cos(\sqrt{\lambda}/2)\cdot\sin(\sqrt{\lambda}x^2/2)\}$$

$$= \alpha \sin\left(\frac{\sqrt{\lambda}}{2}(x^2-1)\right) = \alpha \sin\left(\frac{n\pi}{3}(x^2-1)\right).$$

We may take $\alpha = 1$. Then

$$\lambda_n = \frac{4n^2\pi^2}{9}, \quad F_n(x) = \sin\left(\frac{n\pi}{3}(x^2-1)\right), \quad n = 1, 2, 3\ldots$$

The Sturm-Liouville equation here is

$$\frac{d}{dx}\left[\frac{1}{x}\frac{dF}{dx}\right] = -\lambda x F,$$

so the eigenfunctions should be orthogonal wrt the weight function x. We check this

$$\int_1^2 F_m(x)\, x\, F_n(x)\, dx = \int_1^2 \sin\left(\frac{m\pi}{3}(x^2-1)\right) x \sin\left(\frac{n\pi}{3}(x^2-1)\right) dx$$

Let $u = x^2-1$ in the integral. Then $x\,dx = \frac{du}{2}$

so $\int_1^2 F_m(x)\, x\, F_n(x)\, dx = \frac{1}{2}\int_0^3 \sin\left[\frac{m\pi u}{3}\right]\sin\left[\frac{n\pi u}{3}\right] du$

We know this is zero because we know the functions $\sin\left(\frac{n\pi x}{L}\right)$ are orthogonal on $[0, L]$, and here $L = 3$.

(1) (d) We are now ready to solve the initial value problem. The separated solutions are

$$e^{-\frac{4n^2\pi^2}{9}t}\sin\left[\frac{n\pi(x^2-1)}{3}\right].$$

We superpose these.

$$T(x,t)=\sum_{n=1}^{\infty} C_n e^{-\frac{4n^2\pi^2}{9}t}\sin\left[\frac{n\pi(x^2-1)}{3}\right].$$

We impose the initial condition:

$$1=\sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi(x^2-1)}{3}\right].$$

The functions are orthogonal with respect to x. We multiply by $x\sin\left[\frac{k\pi(x^2-1)}{3}\right]$ and integrate

from 1 to 2 to get

$$C_k=\frac{\int_1^2 x\sin\left[\frac{k\pi(x^2-1)}{3}\right]dx}{\int_1^2 x\left\{\sin\left[\frac{k\pi(x^2-1)}{3}\right]\right\}^2dx}$$

In each integral we make the substitution $u=x^2-1$ to get

$$C_k=\frac{\int_0^3 \sin\left[\frac{k\pi u}{3}\right]du}{\int_0^3 \sin^2\left[\frac{k\pi u}{3}\right]du}=\frac{\frac{3}{k\pi}\left[1-(-1)^k\right]}{\frac{3}{2}}$$

$$=0,\ k \text{ even}$$

$$=\frac{4}{k\pi},\ k \text{ odd}.$$

(1) (d) (continued) In the sum we let $k = 2m+1$ with m taking on all integer values from 0 to infinity. Then

$$T(x,t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} e^{-\frac{4\pi(2m+1)^2 t}{9}} \sin\left[\frac{\pi(2m+1)(x-1)}{3}\right]$$

The exponents in the time term increase rapidly with m, so for large times we can approximate the solution by the first term only:

$$T(x,t) \underset{t\ \mathrm{large}}{\simeq} \frac{4}{\pi} e^{-\frac{4\pi^2 t}{9}} \sin\left[\frac{\pi(x-1)}{3}\right].$$

We see that the dimensionless e-folding time for the first mode is $9/4\pi^2$. The time scale is L^2/D_0, so the dimensional e-folding time is

$$\frac{9L^2}{4\pi^2 D_0}.$$

It is, as we would expect, a dimensionless multiple of L^2/D_0.

See Mathematica notebook for the plots.

CHALLENGE PROBLEM

We have
$$E = \int_0^L \rho C T\,dx = \int_0^L \rho_0 C_0 (1+\tfrac{x}{L}) T_0 \hat{T}(\hat{x},\hat{t})\,dx$$
$$= \int_1^2 \rho_0 C_0 T_0 L\, \hat{x}\hat{T}(\hat{x},\hat{t})\,d\hat{x}$$
$$= \rho_0 C_0 T_0 L \int_1^2 \hat{x}\hat{T}(\hat{x},\hat{t})\,d\hat{x}.$$

(We restore the hats on the dimensionless quantities in this problem.)

CHALLENGE PROBLEM (continued) Using our solution from problem (1), we get

$$E = \rho_0 c_0 T_0 L \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} e^{-\frac{4\pi^2(2m+1)^2}{9}\hat{t}} \int_1^2 \hat{x} \sin\left[\frac{\pi(2m+1)(\hat{x}^2-1)}{3}\right] d\hat{x}$$

The integral is $\int_1^2 \hat{x} \sin\left[\frac{\pi(2m+1)(\hat{x}^2-1)}{3}\right] d\hat{x}$

$$= \int_0^3 \frac{1}{2} \sin\left[\frac{\pi(2m+1)u}{3}\right] du \qquad u = \hat{x}^2 - 1$$

$$= \frac{3}{\pi(2m+1)}$$

So

$$E = \frac{12 \rho_0 c_0 T_0 L}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\frac{4\pi^2(2m+1)^2}{9}\hat{t}}$$

For the special case $t=0$, we have

$$E(0) = \frac{12 \rho_0 c_0 T_0 L}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

We also have $E(0) = \int_0^L \rho_0 c_0 (1 + \frac{x}{L}) T_0 \, dx$

$$= \frac{3 \rho_0 c_0 T_0 L}{2}$$

from which we get the interesting by-product that

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)} = \frac{\pi^2}{8},$$

a sum known to Mathematica. Then

$$\frac{E(t)}{E(0)} = \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\frac{4\pi^2(2m+1)^2}{9}\hat{t}}.$$

See the Mathematica notebook for a plot of this function.

For the fluxes we need $\frac{\partial T}{\partial x}$ which we calculate from our dimensionless solution as

$$\frac{\partial T}{\partial x} = T_0 \frac{\partial \hat{T}}{\partial x} = T_0 \frac{\partial \hat{T}}{\partial \hat{x}} \frac{d\hat{x}}{dx} = \frac{T_0}{L} \frac{\partial \hat{T}}{\partial \hat{x}}.$$

$$= \frac{T_0}{L} \sum_{m=0}^{\infty} \frac{4}{3} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}} 2\hat{x} \cos\left(\frac{\pi[2m+1](\hat{x}^2-1)}{3}\right)$$

and

$$K \frac{\partial T}{\partial x} = \frac{8K_0 T_0}{3L} \sum_{m=0}^{\infty} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}} \cos\left(\frac{\pi[2m+1](\hat{x}^2-1)}{3}\right)$$

so

$$F_{left} = K \frac{\partial T}{\partial x}\Big|_{\hat{x}=1} = \frac{8K_0 T_0}{3L} \sum_{m=0}^{\infty} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}}$$

and

$$F_{right} = -K \frac{\partial T}{\partial x}\Big|_{\hat{x}=2} = \frac{8K_0 T_0}{3L} \sum_{m=0}^{\infty} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}}$$

We see that $F_{left} = F_{right}$, and the total rate of energy flow out is

$$F_{left} + F_{right} = \frac{16 K_0 T_0}{3L} \sum_{m=0}^{\infty} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}}.$$

From the expression for $E(t)$, we get

$$\frac{dE}{dt} = \frac{dE}{d\hat{t}} \frac{d\hat{t}}{dt} = \frac{D_0}{L^2} \frac{dE}{d\hat{t}} = \frac{D_0}{L^2} \frac{12 \rho_0 C_0 T_0 L}{\pi^2} \cdot \frac{d}{d\hat{t}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}}$$

$$= -\frac{D_0}{L^2} \frac{12 \rho_0 C_0 T_0 L}{\pi^2} \frac{5\pi^2}{9} \sum_{m=0}^{\infty} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}}$$

$$= -\frac{16}{3} \frac{K_0 T_0}{L} \sum_{m=0}^{\infty} e^{-\frac{5\pi^2(2m+1)^2\hat{t}}{9}} = -(F_{left} + F_{right})$$

ME 201 / MTH 281 Assignment #6 Solutions Oct. 22, 2009

Problem 1 (d)

We first define the dimensionless solution for *Mathematica*. We call it T[x,t,n], where n is the number of terms to be kept in the partial sum.

$$T[x\_, t\_, n\_] := \sum_{m=0}^{n-1} \frac{4.0}{(2m+1)\pi} \text{Exp}\left[-\frac{4.0\,(2m+1)^2 \pi^2 t}{9}\right] \text{Sin}\left[\frac{(2m+1)\pi\,(x^2-1)}{3.0}\right]$$

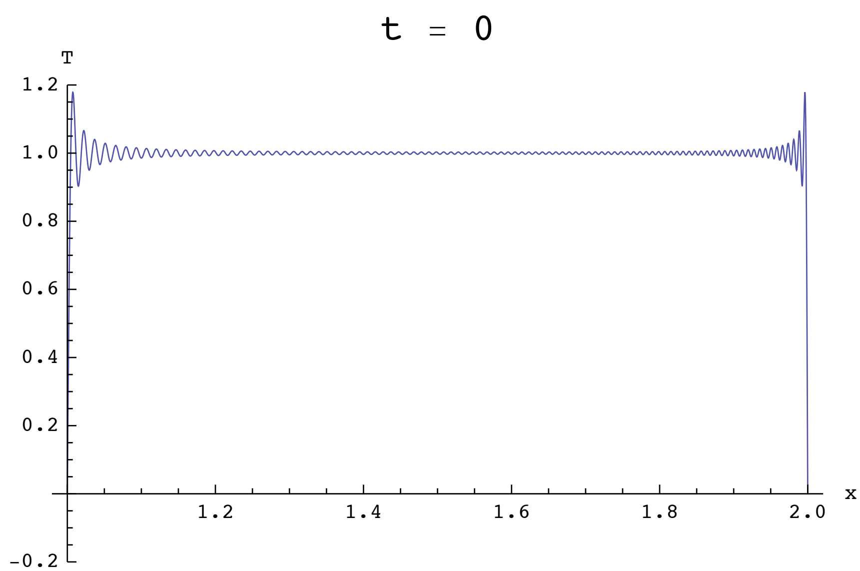
The one term approximation is the first term of this series. We name it Tapprox[x,t].

```
Tapprox[x_, t_] = T[x, t, 1]
```

$$1.27324\, e^{-4.38649\, t}\, \text{Sin}\left[1.0472\,\left(-1+x^2\right)\right]$$

We check our series solution by plotting the initial condition. We will need many terms because of the 1/n convergence, and we expect to see the Gibbs phenomenon. We use 100 terms in the series.

```
Plot[T[x, 0, 100], {x, 1, 2}, AxesLabel → {"x", "T"},
 PlotRange → {-0.2`, 1.2`}, PlotLabel → Row[{"t = ", 0}]]
```



About what we expected. Good convergence over the interior, with the Gibbs overshoots at the endpoints. Now that we know the series is working, we avoid this by defining T at time t=0 to be equal to the initial condition.

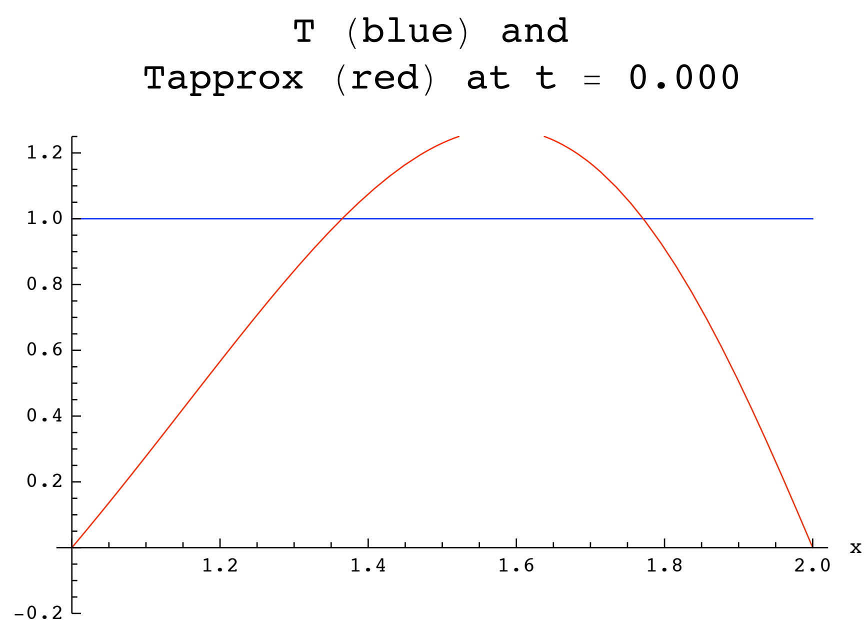
```
T[x_, 0, n_] = 1.0;
```

We now define a function which produces a graph of T versus x for a given time t. We plot T in blue, and the long-time approximation in red. We use 20 terms in the partial sum.

```
graph[t_] := Plot[{T[x, t, 20], Tapprox[x, t]}, {x, 1, 2},
  AxesLabel → {"x", ""}, PlotRange → {-0.2`, 1.25`},
  PlotStyle → {RGBColor[0, 0, 1], RGBColor[1, 0, 0]},
  PlotLabel → Row[{"T (blue) and
Tapprox (red) at t =", PaddedForm[t, {4, 3}]}]]
```

We try this at t = 0.

```
graph[0]
```



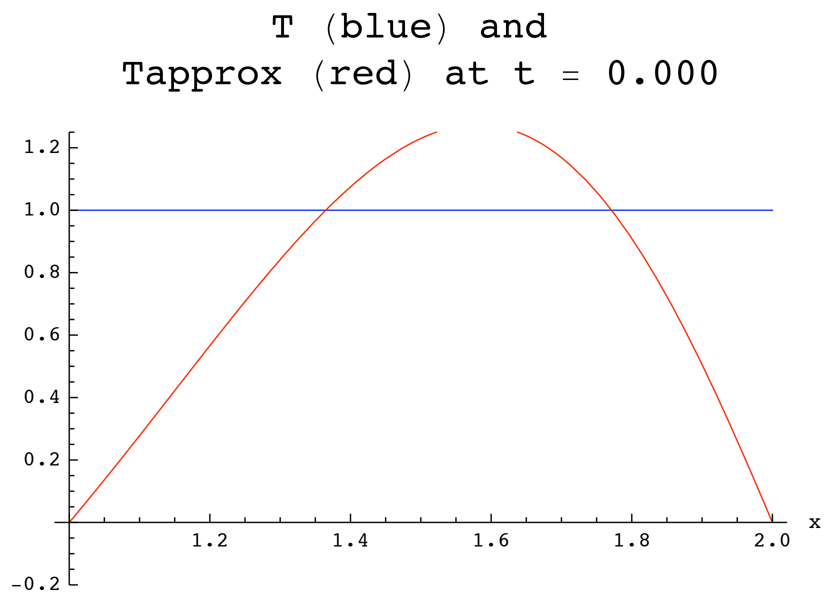
Now we choose the times at which to make a plot. The dimensionless e-folding time of the first mode is

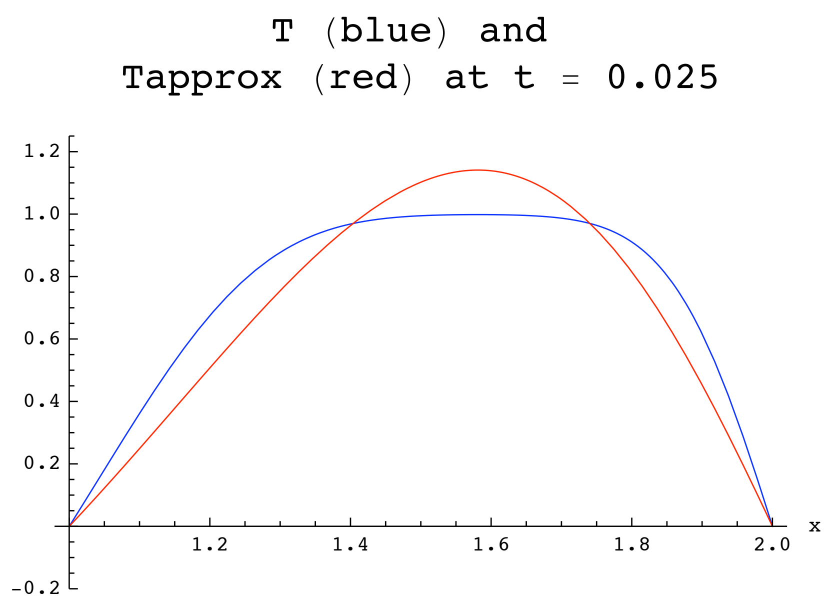
```
9 / (4. π^2)
```

0.227973

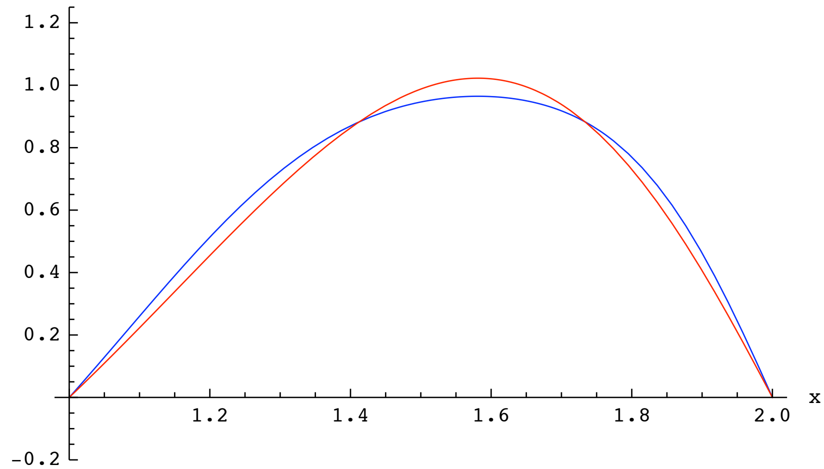
We run our graphs out to t = 0.25. We make 11 graphs at time increments of 0.025.

```
Do[Print[graph[i * 0.025]], {i, 0, 10}];
```

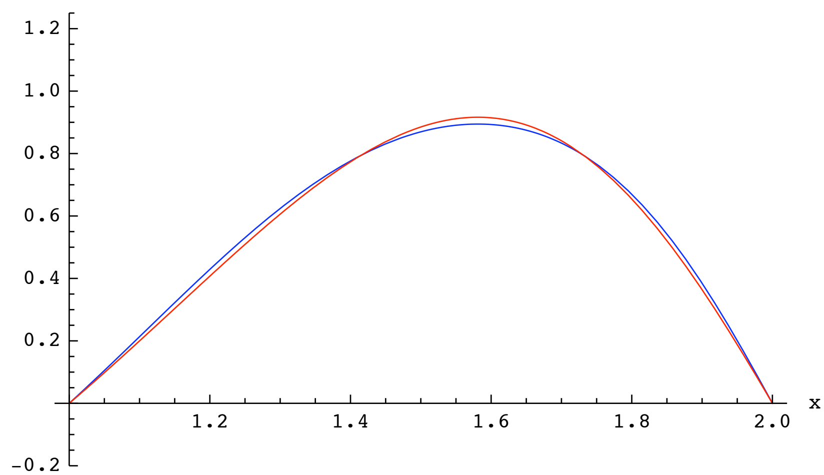

T (blue) and
Tapprox (red) at t = 0.000
1.2
1.0
0.8
0.6
0.4
0.2
-0.2
1.2
1.4
1.6
1.8
2.0
x


T (blue) and
Tapprox (red) at t = 0.025
1.2
1.0
0.8
0.6
0.4
0.2
-0.2
1.2
1.4
1.6
1.8
2.0
x

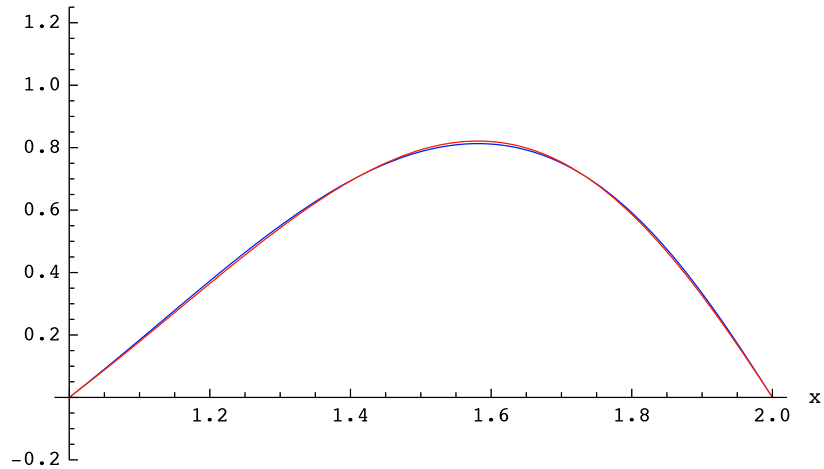
T (blue) and
Tapprox (red) at t = 0.050



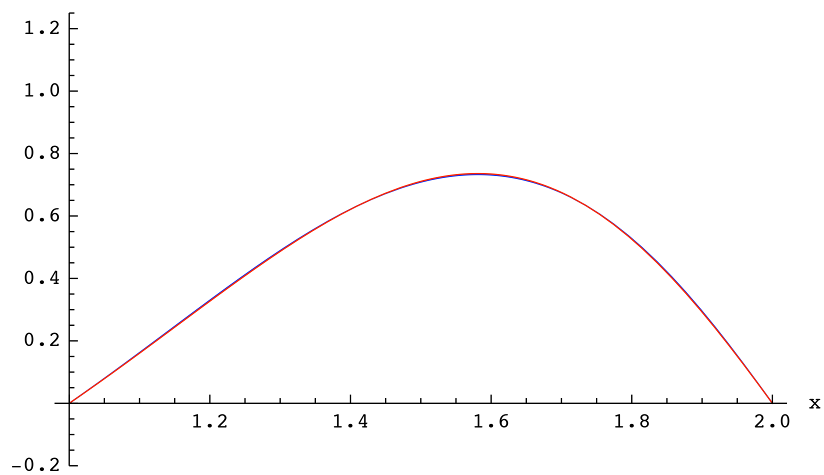
T (blue) and
Tapprox (red) at t = 0.075



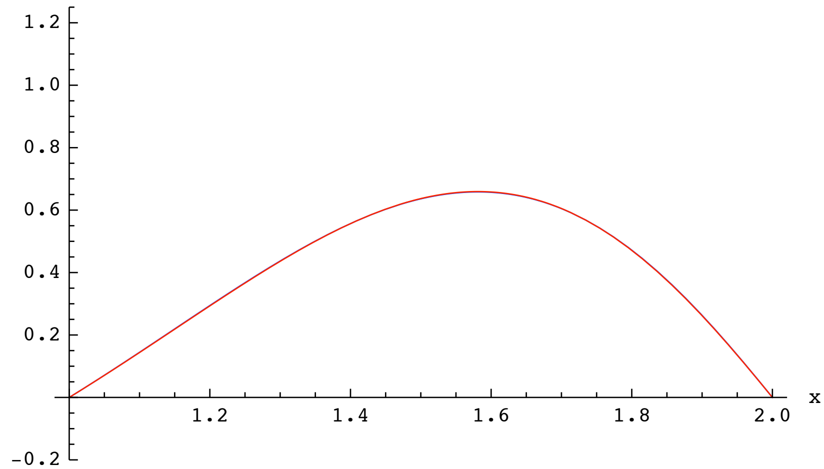
T (blue) and
Tapprox (red) at t = 0.100



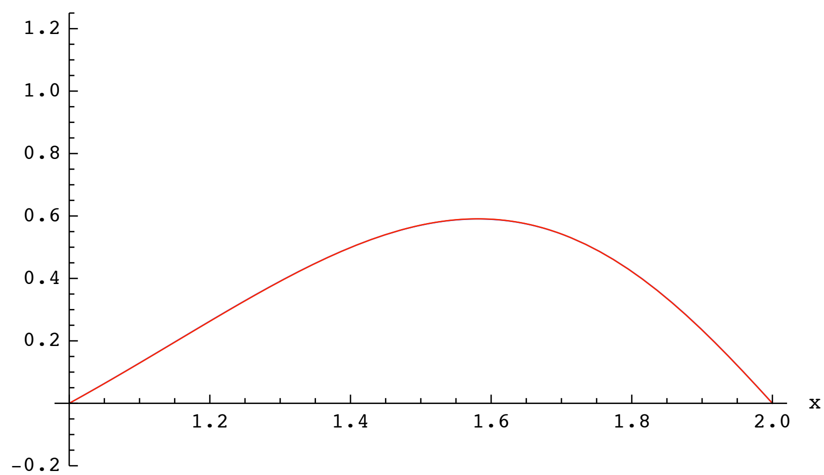
T (blue) and
Tapprox (red) at t = 0.125



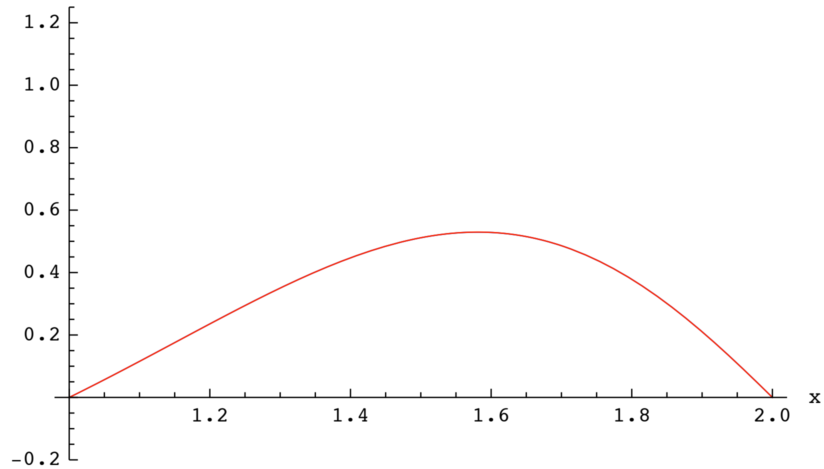
T (blue) and
Tapprox (red) at t = 0.150



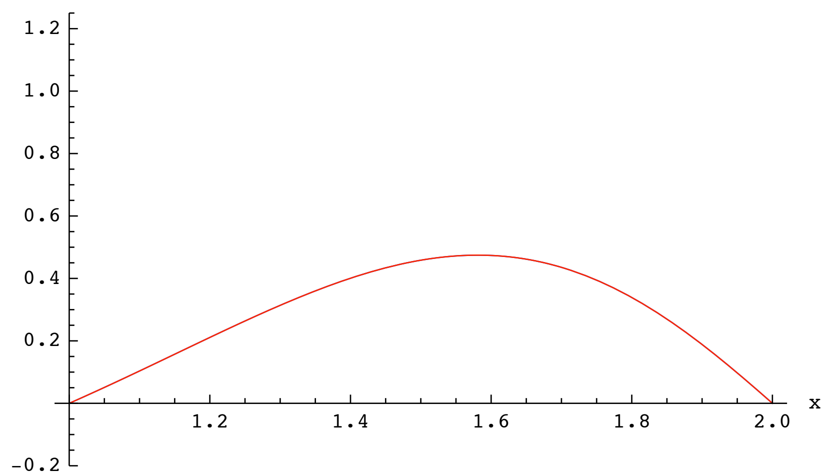
T (blue) and
Tapprox (red) at t = 0.175

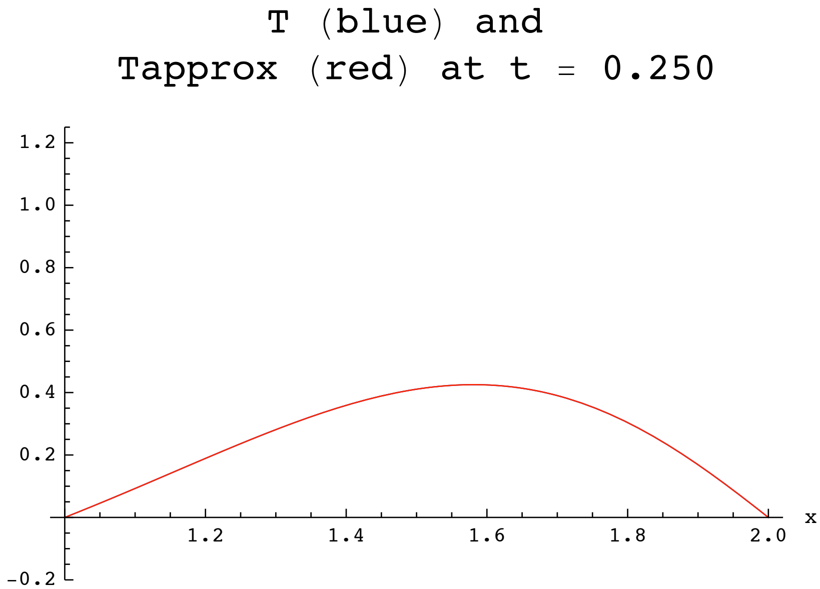


T (blue) and
Tapprox (red) at t = 0.200



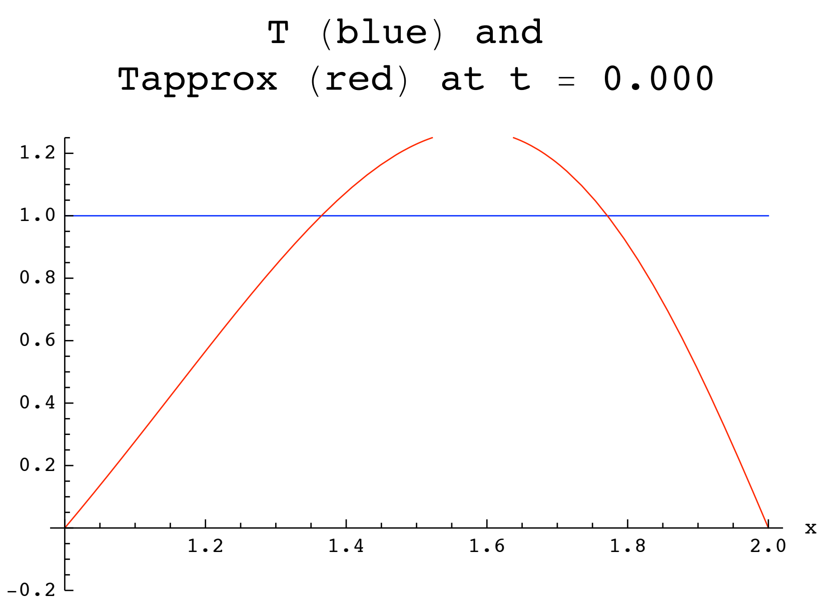
T (blue) and
Tapprox (red) at t = 0.225





For the shorter times, the higher modes are important, but by the time t has reached 0.1, the one term approximation and the series solution essentially coincide graphically. If we make on the order of 100 graphs or so, we can get an excellent movie of the decay process. We do that now, but only show the first graph. The movie is played by going to the Graphics->Rendering->Animate Selected Graphics.

```
Do[Print[graph[i * 0.005]], {i, 0, 100}];
```



■ Challenge Problem

Our task is to plot the ratio of the thermal energy at time t to the initial thermal energy, where we will use the dimensionless time in the plot. As we showed in the solution sheet, this ratio is an infinite sum of time exponentials. We define first a term in the series, and then a partial sum.

```
term[t_, k_] := (1 / (2 k + 1)^2) Exp[- (4 π^2 (2 k + 1)^2 t) / 9]
```

The nth partial sum is

```
energy[t_, n_] := (8 / π^2) Sum[term[t, k], {k, 0, n}]
```

The convergence is slowest at the initial time, but if we take 20 terms, the last term kept will be less than 1/400 of the first term. Thus we may truncate the series at 20 terms with no loss of graphical accuracy. We run the graph out to three times the e - folding time for the first term, hence to

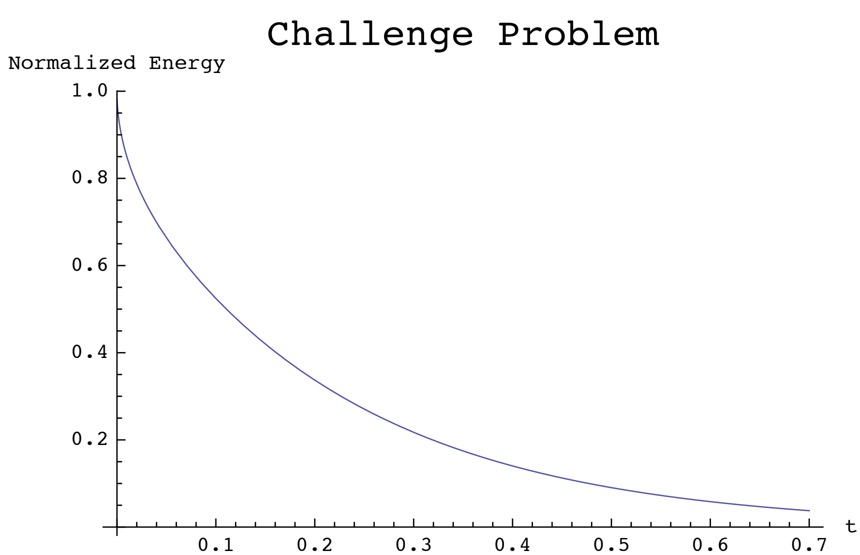
```
(3.0) (9 / (4 π^2))
```

```
0.683918
```

```
tfinal = 0.70
```

```
0.7
```

```
Plot[energy[t, 10], {t, 0, 0.7},
 AxesLabel → {"t", "Normalized Energy"},
 PlotLabel → "Challenge Problem"]
```



No surprises here. A smooth monotonic decay from an initial 1, with significant decay in a moderate multiple of the diffusion time.