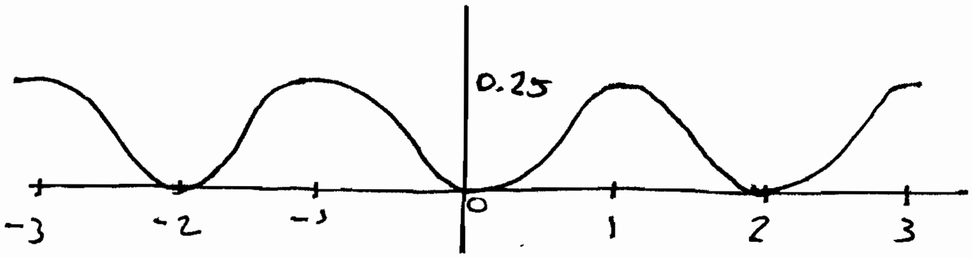
ME 201/MTH 281/ME 400/CHE 400 ASSIGNMENT #3 SOLUTIONS

SEPT 24, 2009

(1) (a) The function is even, and ~~the~~ it vanishes at $x=0$. The slope vanishes at $x=0$ and $x=\pm 1$. The latter two points are maxima at which the function value is $1/4$.



(b) The function and it's derivatives have no discontinuities in the open interval $(-1, 1)$. We check the endpoints to determine the smoothness of the extended function.

$f(-1) = \frac{1}{4} = f(1) \Rightarrow$ extended f continuous

$f'(-1) = 0 = f'(1) \Rightarrow$ extended f' continuous

$f''(-1) = -2 = f''(1) \Rightarrow$ extended f'' continuous

$f'''(-1) = 6 \neq f'''(1) = -6 \Rightarrow$ extended f''' discontinuous.

By our convergence guideline, the Fourier coefficients for f should drop off like $1/n^4$.

(c) We have $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} \frac{1}{4}[2x^2 - x^4]\,dx = \frac{7}{60}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} \frac{1}{4}(2x^2 - x^4)\cos(n\pi x)\,dx$$

$$= \frac{12(-1)^n}{n^4\pi^4}.$$

See the Mathematica notebook for the evaluation of the integrals. This result is consistent with the prediction of part (b).

(2) (a) The function is odd so all of the cosine coefficients vanish. The sine coefficients are

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx = \int_{-1}^{1}(x-x^3)\sin(n\pi x)\,dx$$
$$= 12(-1)^{n+1}/(n^3\pi^3).$$

(See Mathematica notebook for evaluation of the integral.) Then

$$x - x^3 = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3}\sin(n\pi x) \quad \text{on } -1\le x\le 1.$$

(b) In problem (1) we showed that

$$\frac{1}{4}(2x^2 - x^4) = \frac{7}{60} + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x)$$
$$\text{for } -1\le x\le 1.$$

Because of the $1/n^4$ convergence, the series may be differentiated termwise three times. We differentiate the equation above to get

$$x - x^3 = \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3\pi^3}(-\sin(n\pi x))$$

and this is the same series as we got in (a).

(3) We have $f(x) = \frac{1}{4}(2x^2 - x^4)$ and $g(x) = x - x^3$. We also are given that

$$f(x) = \frac{7}{60} + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x) \quad \text{on } -1\le x\le 1,$$

and $$g(x) = \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n^3\pi^3}\sin(n\pi x).$$

We know that $\int_0^x g(x')\,dx' = f(x)$, and we are asked to substitute the series for f and g in this known relation.

(3) (continued)

$$\int_0^x g(x')\,dx' = f(x)$$

$$\sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3}\int_0^x \sin(n\pi x')\,dx' = \frac{7}{60} + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x)$$

$$\sum_{n=1}^{\infty} \frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x')\Big|_0^x = \frac{7}{60} + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x)$$

$$\sum_{n=1}^{\infty} \frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x) - \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^4\pi^4} = \frac{7}{60} + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^4\pi^4}\cos(n\pi x)$$

so $$\sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n^4\pi^4} = \frac{7}{60}$$

or $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

Because the truth of our original equation requires this, we know it is true. Nevertheless we would like some independent confirmation of this novel result. See the Mathematica notebook for such confirmation.

(4) (a) $\underline{E}_1 \cdot \underline{E}_2 = (\underline{i}+\underline{j}+\underline{k})\cdot(-\underline{i}+2\underline{j}-\underline{k}) = -1+2-1 = 0.$
We know the cross-product of two vectors is orthogonal to both factors, so we calculate

$$\begin{aligned} \underline{E}_1\times\underline{E}_2 &= (\underline{i}+\underline{j}+\underline{k})\times(-\underline{i}+2\underline{j}-\underline{k}) \\ &= (-1-2)\underline{i} + (-1+1)\underline{j} + (2+1)\underline{k} \\ &= 3(-\underline{i}+\underline{k}). \end{aligned}$$

Any constant multiple of this will also be orthogonal to $\underline{E}_1$ and $\underline{E}_2$. We take $\underline{E}_3 = -\frac{1}{3}(\underline{E}_1\times\underline{E}_2) = \underline{i}-\underline{k}$.

(4) (continued) (b)

$$\underline{e}_1 = \frac{\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}}, \quad \underline{e}_2 = \frac{-\underline{i}+2\underline{j}-\underline{k}}{\sqrt{6}}, \quad \underline{e}_3 = \frac{\underline{i}-\underline{k}}{\sqrt{2}}$$

(c) $\underline{A} = \underline{i} - 4\underline{j} + \underline{k} = C_1\underline{e}_1 + C_2\underline{e}_2 + C_3\underline{e}_3.$

We dot the equation with $\underline{e}_1$:

$$\underline{e}_1 \cdot \underline{A} = C_1 \underline{e}_1 \cdot \underline{e}_1 = C_1$$

so $C_1 = \dfrac{\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}} \cdot (\underline{i} - 4\underline{j} + \underline{k}) = -\dfrac{2}{\sqrt{3}}$

$$C_2 = \underline{e}_2 \cdot \underline{A} = -\frac{10}{\sqrt{6}}$$

$$C_3 = \underline{e}_3 \cdot \underline{A} = 0$$

Check: $C_1\underline{e}_1 + C_2\underline{e}_2 + C_3\underline{e}_3 = -\dfrac{2}{\sqrt{3}}\dfrac{\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}} - \dfrac{10}{\sqrt{6}}\dfrac{-\underline{i}+2\underline{j}-\underline{k}}{\sqrt{6}}$

$$= \underline{i} - 4\underline{j} + \underline{k}.$$

We have $|\underline{A}|^2 = 1^2 + (-4)^2 + 1^2 = 18$,

and

$$C_1^2 + C_2^2 + C_3^2 = \frac{4}{3} + \frac{100}{6} = \frac{108}{6} = 18.$$

(5) In problem (1) we showed that

$$\frac{1}{4}(2x^2 - x^4) = \underbrace{\frac{7}{60}}_{a_0} + \sum_{n=1}^{\infty} \underbrace{\frac{12(-1)^n}{n^4\pi^4}}_{a_n} \cos(n\pi x)$$

Parseval's theorem says that

$$\int_{-1}^{1} \left[\frac{1}{4}(2x^2 - x^4)\right]^2 dx = 2 \cdot \left(\frac{7}{60}\right)^2 + \sum_{n=1}^{\infty} \left[\frac{12(-1)^n}{n^4\pi^4}\right]^2$$

(5) (continued). The left hand side is

$$\frac{1}{16}\int_{-1}^{1}\left[4x^4-4x^6+x^8\right]dx = \frac{1}{8}\int_0^1\left[4x^4-4x^6+x^8\right]dx$$

$$= \frac{1}{8}\left[\frac{4}{5}-\frac{4}{7}+\frac{1}{9}\right] = \frac{107}{2520}.$$

Then $\displaystyle\frac{144}{\pi^8}\sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{107}{2520} - \frac{49}{1800} = \frac{24}{1575}$

so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{24\,\pi^8}{(1575)(144)} = \frac{\pi^8}{9450}.$

(b) We solve the above expression for π.

$$\pi = \left[9450\sum_{n=1}^{\infty}\frac{1}{n^8}\right]^{\frac{1}{8}}.$$

Keeping only the first term gives 3.14, which is not too bad. We use Mathematica to explore this further.

(c) See Mathematica notebook.

(6) The easiest way to do this problem is to use a favorite mathematician's trick of reducing it to a previous case. Here's what we know. If $h(x)$ has a Fourier series

$$\alpha_0 + \sum_{n=1}^{\infty}\alpha_n\cos(n\pi x/L) + \beta_n\sin(n\pi x/L)$$

on $[-L, L]$, then Parseval's Theorem says that

(6) (continued)

$$\int_{-L}^{L} [h(x)]^2 dx = 2L\alpha_0^2 + L\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2).$$

Now let $h(x) = f(x) + g(x)$, where f and g have Fourier series as given in the problem ~~statem~~ statement. Then

$$\int_{-L}^{L} [h(x)]^2 dx = \int_{-L}^{L} [f(x)]^2 dx + 2\int_{-L}^{L} f(x)g(x)dx + \int_{-L}^{L} [g(x)]^2 dx,$$

and $\alpha_0 = a_0 + A_0$, $\alpha_n = a_n + A_n$, $\beta_n = b_n + B_n$, so we get

$$\int_{-L}^{L} f^2 dx + 2\int_{-L}^{L} fg\, dx + \int_{-L}^{L} g^2 dx$$

$$= 2L(a_0^2 + 2a_0A_0 + A_0^2) + L\sum_{n=1}^{\infty}\{a_n^2 + 2a_nA_n + A_n^2 + b_n^2 + 2b_nB_n + B_n^2\}$$

$$= \left(2La_0^2 + L\sum_{n=1}^{\infty} a_n^2 + b_n^2\right) + 2\left(2La_0A_0 + L\sum a_nA_n + b_nB_n\right) + \left(2LA_0^2 + L\sum_{n=1}^{\infty} A_n^2 + B_n^2\right)$$

Now by Parseval's Theorem applied to f and g, we see that the first terms on each side are equal and the last terms on each side are equal. The remaining terms, after division by 2, give

(5) (continued)

$$\int_{-L}^{L} f(x)g(x)\,dx = 2L a_0 A_0 + L\sum_{n=1}^{\infty} a_n A_n + b_n B_n.$$

An alternative derivation can be based on orthogonality. Calculate $\int_{-L}^{L} f(x)g(x)dx$ by substituting the Fourier series for f and g in the integral. Use orthogonality to simplify the integral of the product of the series, and you get the same result.

As we discussed in class earlier, we may think of $(f, g) = \int_{-L}^{L} f(x)g(x)dx$ as a dot product in the space of functions on $[-L, L]$. The "vectors" are f and g, with components $f(x)$ and $g(x)$ at x. The dot product is then the sum of the products of corresponding components. The right hand side is the dot product calculated in the Fourier representation. In that representation, the vectors components are $\{a_0, a_1, b_1, a_2, b_2 \ldots\}$ and $\{A_0, A_1, B_1, A_2, B_2 \ldots\}$

so the dot product is $a_0 A_0 + a_1 A_1 + b_1 B_1 + \ldots$. This is slightly off because our basis vectors — namely 1, $\cos(n\pi x/L)$, $\sin(n\pi x/L)$ — are not normalized, and this leads to the L and 2L terms in the formula.

CHALLENGE PROBLEM

The mean square error is

$$E = \int_a^b [f(x) - f_{app}(x)]^2 dx = \int_a^b [f(x)]^2 dx$$
$$-2\int_a^b f(x) f_{app}(x)\, dx$$
$$+ \int_a^b [f_{app}(x)]^2 dx .$$

The first term on the right does not depend on the δ's. Let's look more closely at the second term.

$$-2\int_a^b f(x) f_{app}(x)\, dx = -2\sum_{m=1}^{N} \delta_m \underbrace{\int_a^b f(x)\phi_m(x)\, dx}_{= f_m}$$
$$= -2\sum_{m=1}^{N} \delta_m f_m .$$

Now the third term.

$$\int_a^b [f_{app}(x)]^2 dx = \int_a^b \Big[\sum_{m=1}^{N} \delta_m \phi_m(x)\Big]\Big[\sum_{k=1}^{N} \delta_k \phi_k(x)\Big] dx$$
$$= \sum_{\substack{m=1 \\ k=1}}^{N} \delta_m \delta_k \underbrace{\int_a^b \phi_m(x)\phi_k(x)\, dx}_{\substack{= 0 \text{ for } m \neq k \\ = 1 \text{ for } m = k}}$$
$$= \sum_{m=1}^{N} \delta_m^2 .$$

Then $E = \int_a^b [f(x)]^2 dx + \sum_{m=1}^{N} \delta_m(\delta_m - 2f_m)$

We replace $\delta_m(\delta_m - 2f_m)$ by $(\delta_m - f_m)^2 - f_m^2$.

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<u>CHALLENGE PROBLEM</u> (continued)

$$\text{Then}\quad E=\int_a^b [f(x)]^2\,dx-\sum_{m=1}^{N} f_m^2+\sum_{m=1}^{N}(\gamma_m-f_m)^2 .$$

The first two terms are independent of the approximation coefficients γ. The last term is obviously smallest when $\gamma_m=f_m$ for all m. Thus among all possible approximation coefficients, the Fourier coefficients give the smallest mean square error.

An alternative derivation is to set $\frac{\partial E}{\partial \gamma_k}=0$ for $k=1,..N$, and solve the resulting equations for all the γ's.

ME201/MTH281/ME400/CHE400 Assignment #3 Solutions Problems 1, 2, 3, and 5

Problem 1

(c)

We use *Mathematica* to calculate the Fourier coefficients. We have

```
f[x_] := 1/4 (2 x^2 - x^4); L = 1;
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$$f[x\_] := \frac{1}{4}\left(2x^2 - x^4\right);\ L = 1;$$

$$a[0] = \frac{1}{2L}\int_{-L}^{L} f[x]\, dx$$

$$\frac{7}{60}$$

$$a[n\_] = \text{Simplify}\left[\frac{1}{L}\int_{-L}^{L} f[x]\,\text{Cos}[(n\pi x)/L]\, dx,\ n \in \text{Integers}\right]$$

$$\frac{12\,(-1)^n}{n^4\,\pi^4}$$

Problem 2

(a)

$$g[x\_] := x - x^3;\ L = 1;$$

$$b[n\_] = \text{Simplify}\left[\frac{1}{L}\int_{-L}^{L} g[x]\,\text{Sin}[(n\pi x)/L]\, dx,\ n \in \text{Integers}\right]$$

$$-\frac{12\,(-1)^n}{n^3\,\pi^3}$$

Problem 3

We see if Mathematic knows the sum of the series we got from the termwise integration of the equation $g = f'$.

```
Sum[(-1)^(n+1)/n^4, {n, 1, ∞}]
```

$$\frac{7\pi^4}{720}$$

Presumably Mathematica obtained this result from a stored table of known series sums.

Problem 5

(b)

We explore the numerical calculation of π from the formula derived. We define a function which uses k terms in the series to approximate π. We were asked to find the minimum number of terms to reproduce π with the correct first 10 digits to the right of the decimal point: 3.1415926536. We do this by constructing a table of values for $k = 1$ to 30. We keep 11 digits to the right of the decimal point in each answer.

```
piapprox[k_] := (9450 Sum[1/n^8, {n, 1, k}])^(1/8)

Table[Row[{"k = ", k, " π≈", N[piapprox[k], 11]}], {k, 1, 30}]
```

```
{k = 1 π≈3.1399951413, k = 2 π≈3.1415257282, k = 3 π≈3.1415853436,
 k = 4 π≈3.1415913115, k = 5 π≈3.1415923127, k = 6 π≈3.1415925455,
 k = 7 π≈3.1415926134, k = 8 π≈3.1415926367, k = 9 π≈3.1415926458,
 k = 10 π≈3.1415926497, k = 11 π≈3.1415926515, k = 12 π≈3.1415926524,
 k = 13 π≈3.1415926529, k = 14 π≈3.1415926532, k = 15 π≈3.1415926533,
 k = 16 π≈3.1415926534, k = 17 π≈3.1415926535, k = 18 π≈3.1415926535,
 k = 19 π≈3.1415926535, k = 20 π≈3.1415926536, k = 21 π≈3.1415926536,
 k = 22 π≈3.1415926536, k = 23 π≈3.1415926536, k = 24 π≈3.1415926536,
 k = 25 π≈3.1415926536, k = 26 π≈3.1415926536, k = 27 π≈3.1415926536,
 k = 28 π≈3.1415926536, k = 29 π≈3.1415926536, k = 30 π≈3.1415926536}
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We see that 20 terms is sufficient to give 10 correct digits to the right of the decimal point. Here are a few more results. To get 3.14, we need one term. To get 3.1416 we need three terms. To get 3.14159 we need three terms. To get 3.141593 we need six terms.

(c)

```
Sum[1/n^8, {n, 1, ∞}]
```

$$\frac{\pi^8}{9450}$$