ME 201/MTH 281/ME 400/CHE 400 ASSIGNMENT # 1 SOLUTIONS 9/10/09

(1) $\underline{H} = (2x-y)\underline{i} + (z-x-4y)\underline{j} + (y+2z)\underline{k}$

(a) $\nabla \cdot \underline{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2-4+2=0.$

(b) $\nabla \times \underline{H} = \left(\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y}\right)\underline{i} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\underline{j} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\underline{k}$

$= (1-1)\underline{i} + (0-0)\underline{j} + (-1+1)\underline{k} = 0$

We seek Φ such that $\underline{H} = \nabla \Phi$. We start with $\frac{\partial \Phi}{\partial x} = H_x = 2x-y$. We integrate with respect to x to get $\Phi = x^2 - xy + f(y,z)$. We substitute this into $\frac{\partial \Phi}{\partial y} = H_y$ to get $-x + \frac{\partial f}{\partial y} = z - x - 4y$.

We integrate with respect to y to get $f = yz - 2y^2 + g(z)$. We substitute this into $\frac{\partial \Phi}{\partial z} = H_z$ to get

$y + \frac{dg}{dz} = y + 2z$, so $g(z) = z^2 + \text{const}$. Thus Φ is determined to within a constant to be

$$\Phi = x^2 - xy + yz - 2y^2 + z^2.$$

(c) Because $\underline{H} = \nabla \Phi$ and because $\nabla \cdot \underline{H} = 0$, we know that $\nabla \cdot \underline{H} = \nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0$. We can also show this directly. In rectangular coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

so

$$\nabla^2 \Phi = 2 - 4 + 2 = 0.$$

(2) We use the divergence theorem, ~~wh~~ which tells us that

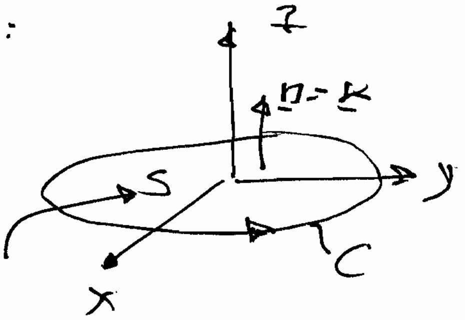
$$\oiint_S \underline{M} \cdot \underline{n}\, d\sigma = \iiint_V \nabla \cdot \underline{M}\, d\tau$$

where V is the volume of the sphere.

$\nabla \cdot \underline{M} = \nabla \cdot \underline{H} + \nabla \cdot (x\underline{i}) = 0 + 1 = 1$, hence $\oiint = \frac{4}{3}\pi (2)^3 = \frac{32\pi}{3}.$

(3) We use Stokes's theorem:

$$\oint_C \underline{G} \cdot d\underline{s} = \iint_S \nabla \times \underline{G} \cdot \underline{n}\, d\sigma$$



We have

$$\nabla \times \underline{G} = \nabla \times (z\underline{H})$$
$$= z\nabla \times \underline{H} + \nabla z \times \underline{H}$$
$$= 0 + \underline{k} \times \underline{H}.$$

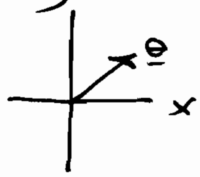
We also know that $\underline{n} = \underline{k}$, so $\nabla \times \underline{G} \cdot \underline{n} = (\underline{k} \times \underline{H}) \cdot \underline{k} = 0$.

$$\therefore \oint_C \underline{G} \cdot d\underline{s} = 0.$$

(4) (a) The temperature gradient is $\nabla T = a\underline{i} + b\underline{j}$. The directional derivative $\frac{dT}{ds}$ in a direction defined by a unit vector $\underline{e}$ is $\frac{dT}{ds} = \underline{e} \cdot \nabla T$.

For a northeast unit vector

$$\underline{e} = \frac{\underline{i} + \underline{j}}{\sqrt{2}}, \text{ so } \frac{dT}{ds} = \frac{\underline{i} + \underline{j}}{\sqrt{2}} \cdot (a\underline{i} + b\underline{j})$$



$$= \frac{a+b}{\sqrt{2}} = \frac{30-20}{\sqrt{2}} = 5\sqrt{2} = 7.07\ ^\circ C/km.$$

As our calculation shows, this has the same value for all x and y.

(b) If we denote the biker's speed by $\frac{ds}{dt}$, then

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt}, \text{ so } \frac{ds}{dt} = \frac{dT/dt}{dT/ds}$$

$$\text{or } \frac{ds}{dt} = \frac{0.5^\circ C/\text{minute}}{7.07^\circ C/km} = \frac{0.5}{7.07}\frac{km}{\text{minute}} = \frac{0.5 \times 1000}{7.07 \times 60}\frac{m}{s}$$

$$= 1.18\ m/s$$

(c) $\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt}$, where $\frac{dT}{ds} = \frac{-\underline{i} + \underline{j}}{\sqrt{2}} \cdot (a\underline{i} + b\underline{j}) = \frac{b-a}{\sqrt{2}} = -\frac{50}{\sqrt{2}}\ \frac{^\circ C}{km}$

(4) (c) (continued)

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \left(-\frac{50}{\sqrt{2}}\ \frac{°C}{km}\right)\left(1.18\times10^{-3}\ \frac{km}{s}\right)\frac{60\,s}{minute}$$

$$= -2.5\ °C/\text{minute}.$$

(5) We use an integrating factor $\mu = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$:

$$t\frac{dx}{dt} + x = t, \quad \text{so} \quad \frac{d}{dt}(xt) = t$$

$$xt = \frac{t^2}{2} + C$$

$$x = \frac{t}{2} + \frac{C}{t}$$

$$x(1) = 1 = \frac{1}{2} + C, \text{ so } C = \frac{1}{2} \text{ and}$$

$$x(t) = \frac{1}{2}\left(t + \frac{1}{t}\right).$$

(6) The equation is constant coefficient, linear and homogeneous, so we try $x = e^{rt}$. Substitution of this into the differential equation gives $r^2 + 2r + 10 = 0$, which has roots $r = -1 \pm 3i$. A real solution basis for these complex conjugate roots is $e^{-t}\cos 3t$ and $e^{-t}\sin 3t$. So

$$x(t) = Ae^{-t}\cos 3t + Be^{-t}\sin 3t.$$

We impose the initial conditions.

$$x(0) = 0 = A$$

$$x'(0) = 3 = B[3] \quad \text{so} \quad B = 1$$

$$\therefore x(t) = e^{-t}\sin 3t$$

(7) We try $y = e^{rx}$ which gives $r^2 + 16 = 0$, so $r = \pm 4i$. The solution basis is $\cos 4x$, $\sin 4x$. So $y(x) = A\cos 4x + B\sin 4x$. The initial conditions give $y(0) = 0 = A$, $y'(0) = 8 = 4B$ so $B = 2$, and

$$y(x) = 2\sin 4x.$$

(8) We try $y = e^{rx}$ to get $r^2 - 4 = 0$, $r = \pm 2$. A solution basis is then e^{2x}, e^{-2x}, so we take
$$y(x) = Ae^{2x} + Be^{-2x}.$$
We impose the initial conditions: $y(0) = 0 = A + B$, and $y'(0) = 2 = 2(A - B)$, hence $A = \frac{1}{2}$, $B = -\frac{1}{2}$ and
$$y(x) = \frac{1}{2}(e^{2x} - e^{-2x}).$$

The hyperbolic functions are a more ~~(convenient~~ convenient basis here: $r = \pm 2$, so a basis is $\sinh 2x$, $\cosh 2x$. We take
$$y(x) = C\sinh 2x + D\cosh 2x.$$
Then $y(0) = 0 = D$, $y'(0) = 2 = 2C$, so $D = 0$, $C = 1$

and $y(x) = \sinh 2x$, which is the same as our earlier solution.

CHALLENGE PROBLEM

(a) We attack the problem directly by imposing the boundary conditions on the general solution. For $\lambda > 0$, the general solution is
$$y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x).$$

From basic differential equation theory, we know that every solution has this form. We impose the boundary condition at $x = 0$:
$$y(0) = 1 = A.$$

Now we impose the condition at $x = 1$:
$$y'(1) = 0 = \sqrt{\lambda}\,(-A\sin\sqrt{\lambda} + B\cos\sqrt{\lambda})$$
so $B = \tan(\sqrt{\lambda})$.

CHALLENGE PROBLEM (continued)

Then $y(x) = \cos(\sqrt{\lambda}\, x) + \tan(\sqrt{\lambda}) \sin(\sqrt{\lambda}\, x)$.

This is a valid solution except for those values of λ for which the tan becomes infinite — equivalently, those values of λ for which $\cos(\sqrt{\lambda})$ vanishes. For positive arguments, the cosine vanishes at $(n+\frac{1}{2})\pi$, where $n = 0, 1, 2, \ldots$. Thus our problem does not have a solution for

$$\lambda = \lambda_n = (n+\tfrac{1}{2})^2 \pi^2, \quad n = 0, 1, 2 \ldots$$

(b) We repeat the analysis of (a), now using $y(0)=0$ and $y'(1)=0$. The general solution is

$$y(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x).$$

The boundary condition at $x=0$ gives

$$y(0) = 0 = A,$$

and the boundary condition at $x=1$ gives

$$y'(1) = 0 = B\sqrt{\lambda}\cos(\sqrt{\lambda}).$$

Because $\lambda > 0$, this requires either $B=0$ or $\cos(\sqrt{\lambda})=0$. If $B=0$, we get the trivial solution $y \equiv 0$. For certain values of λ, $\cos(\sqrt{\lambda})=0$, and we get a non-trivial solution. $\cos(\sqrt{\lambda})=0$ whenever $\sqrt{\lambda} = (n+\frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$ So there are non-trivial solutions whenever

$$\lambda = \lambda_n = (n+\tfrac{1}{2})^2 \pi^2, \quad n = 0, 1, 2, \ldots$$

As we shall see later, it is not a coincidence that questions (a) and (b) gave the same answer.