

ME 201/MTH 281/ME 400/CHE 400

ASSIGNMENT #8 2009

Assignments handed in by 6 PM on Wednesday Nov. 4 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday Nov. 5 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Nov. 5. This is the last homework assignment before Exam #2 on Nov. 12.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
V. SEPARATION OF VARIABLES, PART 2		
5.4 Organ Pipe Acoustics	W Oct 28	---
5.5 Standing Acoustic Modes in Three Dimensions	Th Oct 29	7.3,7.4
VI. UNBOUNDED DOMAINS		
6.1 Fourier Integral	F Oct 30	10.1-10.3
6.2 Fourier Transform	M Nov 2	10.3,10.4

PROBLEMS

(1) (15 points) A stretched string occupies the region $0 \leq x \leq L$. The end at $x = 0$ is fixed so that the displacement is zero there. The end at $x = L$ is attached to a frictionless and massless slider, which slides freely in the vertical direction, so that the slope $\partial y / \partial x$ is zero there. The equation and boundary conditions are

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, 0 < x < L \text{ and } t > 0, \text{ with } y(0,t) = 0 \text{ and } \frac{\partial y}{\partial x}(L,t) = 0.$$

Find the frequencies of vibration of the normal modes of the string. Give your answers in terms of the linear frequencies in Hz.

(2) (15 points) Find the length required for a pipe organ pipe open at both ends to produce a fundamental tone two octaves below middle C (the frequency is 65.41 Hz). Base your calculations on air at 20 °C. What will the fundamental frequency be on a cold winter day when the temperature in the poorly heated building containing the organ has dropped to 5 °C? Is this a significant change in frequency?

(3) (30 points) Consider a vibrating membrane which has a tension T (force per unit length), and a mass per unit area σ , and suppose that there is an elastic restoring force which tends to return each point of the membrane to equilibrium. Let the vertical displacement of the membrane be $\Psi(x,y,t)$, where x, y are coordinates in the plane of the membrane and t is time. It can be shown that the equation of motion for small displacements in this case is

$$\frac{\partial^2 \Psi}{\partial t^2} = C^2 \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) - \frac{\alpha}{\sigma} \Psi,$$

where $C^2 = T/\sigma$. The last term describes the elastic restoring force with α being like a spring

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(3) (continued) constant for this force. (In other contexts, this equation is known as the Klein-Gordon equation.) The membrane occupies the rectangular region $0 \leq x \leq a$, $0 \leq y \leq b$, and the displacement Ψ is zero everywhere along the boundary of this region.

(a) (20 points) Find the frequencies of the normal modes of this system. Show that for the higher modes, the frequencies are approximately the same as they would be if α were zero.

(b) (10 points) Find the number of modes with frequencies between 2000 and 4000 Hz. Use the following parameter values: $a = b = 0.4$ m, $T = 100$ N/m, $\sigma = 0.1$ kg/m², and $\alpha = 10^3$ N/m³.

(4) (20 points) Consider the function $f(x) = \begin{cases} e^{\alpha x} & \text{for } x < b, \\ 0 & \text{for } x \geq b \end{cases}$ where $a > 0$ and $b < 0$.

(a) (7 points) Find the Fourier transform $\tilde{f}(k)$ of $f(x)$ by directly evaluating the integral defining the transform.

(b) (6 points) Verify that $\int_{-\infty}^{\infty} f(x) dx = \tilde{f}(0)$.

(c) (7 points) Verify Parseval's theorem for this transform pair by directly evaluating both of the relevant integrals.

(5) (20 points) In this problem, you will apply the Fourier Transform to the solution of a simple ordinary differential equation for a function $y(x)$. The equation is

$$y'' - y = e^{-\alpha|x|}.$$

Here α is a positive constant (you may assume that $\alpha \neq 1$), and y must satisfy the following additional conditions (these are the conditions that allow the Fourier transform):

$y \rightarrow 0$ as $x \rightarrow \pm\infty$, and y and y' continuous.

(a) (10 points) Take the Fourier transform of the equation and find $\tilde{y}(k)$, the Fourier transform of $y(x)$.

(b) (10 points) Invert the transform to find $y(x)$. Do this by using partial fractions and then one of the Fourier transform pairs given in class.

CHALLENGE PROBLEM

This is an extension of problem (1). Here you are asked to find the frequencies of the normal modes when the frictionless slider has a finite mass. The equation of motion of the string is still

$$\sigma \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L \text{ and } t > 0,$$

where σ is the mass per unit length of the string, and T is the tension in the string. The string is fixed at the left end, so the boundary condition there is $y(0, t) = 0$. To get the boundary condition at the right end, we must do a force balance on the slider. From the picture below, we

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CHALLENGE PROBLEM (continued)

can see that the vertical force on the slider is $T \sin(\theta)$ where T is the tension in the string. For small slopes we have $\sin(\theta) \approx \tan(\theta) = -\frac{\partial y}{\partial x}(L,t)$, hence the boundary condition at the right end

is $m \frac{\partial^2 y}{\partial t^2}(L,t) = -T \frac{\partial y}{\partial x}(L,t)$, where m is the mass of the slider.

(a) Look for normal modes of the form $y(x,t) = \cos(\omega t)\phi(x)$. Give the complete formulation of the eigenvalue problem for the eigenfunction ϕ and the frequency ω^2 . Explain why this is not a standard Sturm-Liouville eigenvalue problem.

(b) By solving the equation for ϕ and imposing the boundary conditions, derive the equation satisfied by the frequency ω^2 . Use the differential equation for ϕ and the boundary conditions to show that there are no negative values of ω^2 . Use the eigenvalue equation (rearranging it if necessary) and a graph to convince yourself (and me) that there are infinitely many positive values of ω^2 .

(c) Find the frequencies for the first three modes for the following parameter values: $L = 0.5$ m, $\sigma = 4 \times 10^{-3}$ kg/m, $T = 40$ N, and $m = 2 \times 10^{-3}$ kg.

