ME 201/MTH 281/ME 400/CHE 400
ASSIGNMENT #7 2009

Assignments handed in by 6 PM on Wednesday Oct. 28 will receive a 5-point bonus. Assignments handed in after that but by 6 PM on Thursday Oct. 29 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
V. SEPARATION OF VARIABLES, PART 2		
5.1 Heat Conduction with Newton's Law of Cooling	Th Oct 22	5.8
5.2 Heat Conduction with Sources	F Oct 23	8.3
5.3 Critical Size of a Nuclear Reactor	M Oct 26	---

PROBLEMS

(1) (40 points) The object of this assignment is to make use of the Mathematica notebook entitled "Newton's Law of Cooling" to solve a heat conduction problem very similar to the one solved in class in section 5.1 of the notes. The notebook will be handed out in class on Thursday Oct. 22, and is available on the web now. Your task is to modify that notebook so that it solves the following problem:

$$\frac{\partial T}{\partial t} = D_f \frac{\partial^2 T}{\partial x^2}, 0 < x < L, t > 0,$$

$$\text{with } T(0,t) = T_L,\ k\frac{\partial T}{\partial x}(L,t) + h[T(L,t) - T_A] = 0,$$

$$\text{and } T(x,0) = T_0.$$

The values of the constants and parameters in the problem are

$$h = 22.4 \text{ W/m}^2 \cdot \text{K},$$
$$L = 0.5 \text{ m},$$
$$k = 2.80 \text{ W/m} \cdot \text{K},$$
$$D_f = 1.37 \times 10^{-6} \text{ m}^2/\text{s},$$
$$T_L = 0^\circ\text{C}, T_A = 10^\circ\text{C}, T_0 = 0^\circ\text{C}.$$

You should pattern your solution after the given notebook. Your solution should include graphs of the temperature versus *x* for a number of different times. Use your solution to find a long-time approximation in which one term of the series for the transient is kept, and compare graphically your approximation with the full solution.

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(2) (40 points) Consider the boundary value problem given below, which describes the transient heat flow in a slab of width L produced by a heat source γ which is constant in space and time.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} + \gamma,\ 0 < x < L,\ t > 0,$$
$$\text{with } T(0,t) = 0,\ T(L,t) = 0, \text{ and } T(x,0) = 0\ .$$

(a) (10 points) Solve this in the usual way by splitting the problem into steady-state and transient parts. Solve for the transient by separation of variables, and for the steady state by two simple integrations.

(b) (20 points) In this part of the problem you are asked to construct the solution in a different way. The method developed here is more general, and will work in cases when the source term depends on both x and t. Proceed by (1) looking for a solution in the form $T(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin\left(n\pi x / L\right)$, (2) expanding the heat source γ in a series of the sine functions, and (3) substituting both expansions into the equation for T. In this way, obtain ordinary differential equations to solve for the coefficients $C_n(t)$. Solve these equations and impose appropriate initial conditions on each C_n, as determined from the initial condition on $T(x,t)$.

(c) (10 points) Verify that the solutions obtained in parts (a) and (b) are the same.

(3) (20 points) A nuclear reactor is made in the shape of a rectangular parallelepiped with a square cross-section. The dimensions are a by a by c. The height c is given. Find the value of a for which the reactor is critical. What is the minimum value of c for which the reactor can be critical? The governing equation (as given in class) is

$$\frac{\partial N}{\partial t} = D\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} + \frac{\partial^2 N}{\partial z^2}\right) + \alpha N - \beta N\ , 0 < x < a\ , 0 < y < a\ , 0 < z < c\ ,$$

where $\alpha > \beta > 0$. The boundary condition is that the neutron density N should vanish on the edges of the rectangular parallelepiped. You may use the spatial modes found in class:

$$\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{a}\right)\sin\left(\frac{r\pi z}{c}\right), \text{ where } p,\ q, \text{ and } r = 1,\ 2,\ 3,\ \cdots\ .$$

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CHALLENGE PROBLEM

This is a continuation of problem 1, in which you are asked to study how the decay time for the first mode varies with the convective heat transfer coefficient *h*. All of the calculations here are for the transient temperature $T_r(x,t)$, which, as you showed in problem 1, is a solution of

$$\frac{\partial T_r}{\partial t} = D_f \frac{\partial^2 T_r}{\partial x^2},\ 0 < x < L,\ t > 0,$$

$$\text{with } T_r(0,t) = 0,\ k\frac{\partial T_r}{\partial x}(L,t) = -hT_r(L,t), \text{ and } T_r(x,0) = T(x,0) - T_s(x),$$

where $T_s(x)$ is the steady-state temperature. As you also showed in problem 1, the separated solutions have the form $e^{-\lambda D_f t}\sin(\sqrt{\lambda}x)$, where $\lambda = (z/L)^2$, with z being a root of $\tan(z) = -z/B_i$, where $B_i = (hL)/k$ is the Biot number.

(a) From the graph of the eigenvalue equation, show that for any positive value of B_i the first root z_1 satisfies $\pi/2 < z_1 < \pi$.

(b) If *h* is very large, the heat transfer from the end of the bar to the surroundings is very good, and we expect the temperature at the end of the bar to be close to the temperature of the surroundings. Use the eigenvalue equation or its graph (or both) to find the limiting value of z_1 as *h* becomes very large.

(c) If *h* is very small, the right end of the bar is very nearly insulated. Reasoning as in part b, find the limiting value of z_1 as *h* goes to zero.

(d) Using the numerical values of the parameters given in problem 1, plot the decay time of the first mode ($1/(\lambda_1 D_f)$) in hours as a function of *h* from *h* = 0 to $h = 100\ \text{W/m}\cdot\text{K}$.